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On the relationship of periodic wavetrains and solitary waves of complex Ginzburg-Landau type equations

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Abstract

The relationship between periodic wavetrains and solitary waves in complex Ginzburg-Landau type equations, such as those that model optical fiber amplifiers, is studied in the nonlinear Schrödinger limit with a Melnikov method. An important example, the cubic complex Ginzburg-Landau equation, is studied in detail. For this equation it is found that in the NLS limit particular families of periodic wavetrains all deform asymptotically to a single, persisting, stationary, nonlinear Schrödinger soliton as their periods tend to infinity. © 1997 Published by Elsevier Science B.V.

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1. Introduction

In standard nondimensional soliton units, for pulse-widths greater than about a picosecond, the evolution of pulses in monomode optical fibers is well described to leading order by the integrable, nonlinear Schrödinger (NLS) equation [1-3]

$$\frac{\partial q}{\partial Z} = \frac{\mathrm{i}}{2}\frac{\partial^2 q}{\partial T^2} + \mathrm{i}|q|^2 q. \qquad (1)$$

Here q is the complex envelope of the electric field, Z is the coordinate along the fiber, and T is the retarded time in a frame moving with the group velocity associated with some carrier frequency ω_0. This equation, which appears also in many other contexts other than fiber optics, supports soliton solutions, i.e. stable pulses which collide elastically with only a change of phase. This fact has inspired intense research over the past decade on the possibility of exploiting solitons for high bit-rate communication systems.

In this Letter we investigate which traveling wave solutions of the NLS equation, both solitary and periodic, "persist", or are "selected" when perturbations describing optical amplification, such as that resulting from erbium doping or Raman gain, are added to the NLS equation. As will be seen from our results, optical amplification destroys most traveling wave solutions except for a small subset.

The persistence of NLS solitary waves has been considered to some extent before, and the persisting solitary waves found are sometimes called "amplifier solitons" or "autosolitons" [4]. For some impor-

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tant special cases (linear gain with Gaussian filtering) exact autosoliton solutions have been found [7–9]. These solutions are stationary in the retarded frame associated with the carrier frequency at the gain peak, which is consistent with the intuition that frequency dependent gain ought to center the spectrum of a pulse around the gain peak. In this Letter we elucidate the relationship of these autosolitons to families of non-stationary, periodic, persisting wavetrains, which are also important in applications. For example, periodic wavetrains of this type are relevant to fiber optical soliton sources for which the fiber dispersion and nonlinearity play a significant role, such as in many fiber lasers [16].

We have studied the persistence of these periodic waves as a subtopic in another series of papers [10–12], which we will refer to here as CLL1, CLL2, and CLL3. Besides the relationship of these waves to the solitary waves, the inclusion of a physically relevant type of nonlocal perturbation, with which optical amplifiers such as erbium-doped fiber amplifiers (EDFAs) can be modeled, is also considered here.

The rest of the Letter is organized as follows. In Section 2 we precisely pose the problem to be analyzed. In Section 3 we derive Melnikov conditions for the persistence of traveling waves. In Section 4 we briefly summarize the relevant results found in CLL1–CLL3 for periodic waves, focusing especially on the cubic complex Ginzburg–Landau (CGL) equation. Then, in Section 5, we relate these results to the persisting autosolitons.

2. Problem definition

We study the perturbed NLS equation

$$\frac{\partial q}{\partial Z} = \frac{\mathrm{i}}{2}\frac{\partial^2 q}{\partial T^2} + \mathrm{i}|q|^2 q + \epsilon\left[D_g\frac{\partial^2 q}{\partial T^2} - (g'(|q|^2) + h'(I(q)))q\right], \qquad (2)$$

which contains in square brackets a very general Ginzburg–Landau type perturbation that models optical amplification processes. The physical meaning of each of these terms is explained further below. The parameter ϵ is introduced to control the size of the perturbation for the mathematical analysis. The functions g and h, whose derivatives appear in the perturbation, are real analytic functions on $[0,\infty)$ with at most polynomial growth. The point of view we are taking is that the perturbation ultimately of interest for applications is obtained by setting $\epsilon = 1$, but that the main obstruction for the existence of traveling waves is their persistence under small perturbation.

The diffusion term in the perturbation, with diffusion coefficient D_g, models the filtering effect of band-limited gain, and is derived under the assumptions that the spectrum of the waves under consideration have a spectrum narrower than the gain spectrum (which is true for many fiber-optic applications), and that the reference carrier frequency ω_0 around which the perturbed NLS envelope equation has been derived is equal to the frequency of a peak in the gain spectrum. The latter assumption implies that the retarded frame in which we work has an absolute physical reference.

The term containing the function $g'(\cdot)$ models both linear gain and fast saturation effects such as two-photon absorption [5,6,9]. For example, in Sections 4 and 5, we specialize to

$$g'(z) = -R + sz, \quad h'(z) = 0, \qquad (3)$$

which yields the ubiquitous cubic complex Ginzburg–Landau (CGL) equation

$$\frac{\partial q}{\partial Z} = \frac{\mathrm{i}}{2}\frac{\partial^2 q}{\partial T^2} + \mathrm{i}|q|^2 q + \epsilon\left(D_g\frac{\partial^2 q}{\partial T^2} + Rq - s|q|^2 q\right). \qquad (4)$$

Various versions of this equation have been used to model fiber optical amplification before [8,9,4].

To model EDFAs, which saturate on time scales (milliseconds) much longer than a typical pulse (picoseconds) in fiber optical applications, we have also included the term $h'(I(q))$ in the perturbation which depends only on the average intensity

$$I(q) = \frac{1}{T_0}\int_0^{T_0} \mathrm{d}T|q|^2, \qquad (5)$$

where T_0 is the fundamental period of the wavetrain. Nonlocal terms of this type were not considered in CLL1–CLL3. For a solitary pulse, $T_0 = \infty$, $I(q)$ is identically zero, and only the first two terms of the perturbation are present in this case. As far as the persistence of traveling waves is concerned, the results obtained with a term $h'(I(q))$ in the perturbation are very similar to those obtained with a g term, so later on we will only consider a detailed example with a g.

Having defined the equations to be studied, we now describe the traveling wave solutions of the unperturbed NLS equation which we test for persistence. Here we focus on what we called in CLL2 the "uncentered" NLS traveling waves. We call these uncentered because their orbits in the potential governing the NLS traveling waves are not symmetrical about the origin (see CLL2 for details). These waves have amplitude everywhere greater than zero, and form the bulk of the NLS traveling wavetrains in a parametric sense, and are the wavetrains most relevant to telecommunication oriented fiber optics. There is also a smaller class of waves we named "centered waves" in CLL2 which we do not discuss here because the persistence properties of these wavetrains and their relationship to the solitary waves is identical to the case of the uncentered waves with $\delta = 1$ (δ is defined below).

All of the uncentered periodic traveling waves of the unperturbed cubic NLS equation ($\epsilon = 0$) of period T_0 have the form

$$q = Q(\xi)\,\mathrm{e}^{-\mathrm{i}\alpha Z + \mathrm{i}S(\xi)}, \tag{6}$$

where $\xi = T - cZ - \theta_0$, and where

$$Q(\xi) = \eta\sqrt{\mathrm{dn}^2(\eta\xi,\kappa) - 1 + \delta^2}, \tag{7}$$

$$S(\xi) = \int_0^{\xi}\left(c + \frac{\mu}{Q^2(\xi')}\right)\mathrm{d}\xi' + \sigma_0, \tag{8}$$

and

$$c = \frac{1}{T_0}(2\pi n - 2m\mu\Pi(\kappa, -\kappa^2/\delta^2) \times \sqrt{(1-\delta^2)(1-\kappa^2/\delta^2)}), \tag{9}$$

$$\mu = \pm\eta^3\sqrt{\delta^2(1-\delta^2)(\delta^2-\kappa^2)}, \tag{10}$$

$$\alpha = \frac{\eta^2}{2}(3(1-\delta^2) - (2-\kappa^2)) - \frac{c^2}{2}. \tag{11}$$

Some examples of these waves can be seen in Fig. 6 at the end of the Letter. The function $\mathrm{dn}(\cdot,\kappa)$ is the Jacobi elliptic dnoidal function with modulus κ, $\Pi(\kappa, b)$ is the complete elliptic integral of the third kind, and κ is related to T_0 and η through the periodicity constraint

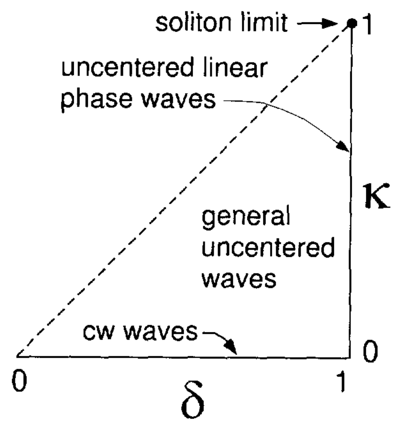

Fig. 1. The κ, δ simplex.

$$2mK(\kappa) = T_0\eta. \tag{12}$$

Here, $K(\kappa)$ is the complete elliptic integral of the first kind, which entails that $0 \leqslant \kappa < 1$. Due to the fact that the period of the Jacobi dnoidal function $\mathrm{dn}^2(\cdot,\kappa)$ is $2K(\kappa)$, the parameter m is a positive integer equal to the number of oscillations of the amplitude in a complete period T_0. We therefore call m the "modulation number". The parameter δ can be any real number such that $\kappa < \delta \leqslant 1$. The parameter n is a non-negative integer and is the winding number of the phase over one period. The parameters σ_0, and θ_0 are trivial constants of motion related to the S^1 phase symmetry and translational invariance of the NLS equation, respectively.

In practice it is easiest to think of these solutions as consisting of families indexed by the integers n and m, and the two signs of μ. Then, the solutions in each family, for a fixed value of T_0, are parameterized by the continuous parameters κ and δ lying within the right triangular "simplex" (see Fig. 1) defined by

$$0 \leqslant \kappa < \delta \leqslant 1. \tag{13}$$

For any point in the simplex, for given choices of n, m, and T_0, the parameter η is uniquely determined by the constraint (12). Fig. 1 also indicates the locus of certain special subfamilies of solutions found at three extremes of the simplex. We now describe these subfamilies, as these will play important roles later on.

At the lower edge of the simplex ($\kappa = 0$), the traveling waves reduce to continuous wave (cw) solutions (sometimes called "rotating waves" in the mathematical literature)

$$q^{\text{cw}} = \frac{\pi}{T_0} m\delta \times \exp\left[\mathrm{i}\left(\frac{2\pi n}{T_0} T - \frac{1}{2}\left(\frac{2\pi n}{T_0}\right)^2 Z + \left(\frac{\pi}{T_0} m\delta\right)^2 Z \right)\right]. \tag{14}$$

In fact, these solutions exist for the NLS equation for any choice of $\delta > 0$. It is shown in CLL2 that the subset of cw waves along the lower edge of the simplex are exactly the cw waves which are linearly stable in the *unperturbed* NLS equation to sideband perturbations of the form

$$q = q^{\text{cw}}(1 + p_+(Z)\,\mathrm{e}^{\mathrm{i}k_m T} + p_-(Z)\,\mathrm{e}^{-\mathrm{i}k_m T}), \tag{15}$$

where $k_m = 2\pi m/T_0$ is the wavenumber of the perturbation and $p^{\pm}(Z)$ are the (small magnitude) complex perturbation amplitudes. The cw waves with $\delta > 1$ are all linearly unstable to wavenumber k_m perturbations. The cw wave solutions with $\delta \leqslant 1$ are degenerate in the sense that all traveling wave families having the same period and winding number n but different values of the modulation number m all reduce to the same family of cw waves.

At the right-hand edge of the simplex ($\delta = 1$) are located traveling waves with linear phase profiles, which we call "uncentered linear phase waves" of the form

$$q = \eta\,\mathrm{dn}(\eta[T - cZ - \theta_0]) \times \exp[\mathrm{i}(cT + \tfrac{1}{2}[\eta^2(2-\kappa^2) - c^2]Z + \sigma_0)], \tag{16}$$

where $c = 2\pi n/T_0$ and $\eta = 2m\,K(\kappa)/T_0$.

Finally, we may take a soliton (solitary wave) limit of the NLS traveling wave solutions by taking $T_0 \to \infty$. This limit can be taken with η and c fixed, which forces $\kappa \to 1$ and $\delta \to 1$ along paths in the simplex which depend on T_0, n, m, η, and c, and which asymptotically approach the upper right-hand corner of the simplex. When the limit is taken in this way, the traveling waves asymptotically have the form of a chain of one-solitons given by

$$q(Z,T) = \sum_{j=-\infty}^{j=+\infty} \eta\,\mathrm{sech}\left(\eta\left[T - cZ - j\frac{T_0}{m} - \theta_0\right]\right) \times \exp\left(\mathrm{i}cT + \frac{\mathrm{i}}{2}(\eta^2 - c^2)Z + \frac{2\pi n}{m} j + \mathrm{i}\sigma_0\right) + \mathrm{O}(1/T_0), \tag{17}$$

where η, κ, θ_0, and σ_0 are the (real) soliton parameters.

Our goal is to determine the persistence of all the above solutions when ϵ is made different from zero.

3. Melnikov conditions

In this section we derive the Melnikov conditions for the persistence of traveling wave solutions when ϵ is made different from zero in Eq. (2). Our derivation works directly from the governing partial differential equation, but yields conditions which are precisely equivalent to the conditions derived via the classical Melnikov method from the ODEs governing the traveling waves. This equivalence is shown in CLL2. The derivation here is slightly simpler than that given in CLL2.

We begin by writing Eq. (2) in the variational form

$$\partial_Z q = -\mathrm{i}\frac{\delta H}{\delta q^*} - \epsilon\frac{\delta G}{\delta q^*}, \tag{18}$$

where H is the NLS Hamiltonian

$$H \equiv \tfrac{1}{2}\int_0^{T_0} \left(|\partial_T q|^2 - |q|^4\right)\,\mathrm{d}T, \tag{19}$$

$$G \equiv \int_0^{T_0} \left(D_g|\partial_T q|^2 + g(|q|^2)\right)\,\mathrm{d}T + T_0 h(I(q)). \tag{20}$$

For any functional F that is conserved by the unperturbed NLS equation, we have

$$\frac{\mathrm{d}F}{\mathrm{d}Z} = -\epsilon\int_0^{T_0}\left(\frac{\delta F}{\delta q^*}\frac{\delta G}{\delta q} + \frac{\delta F}{\delta q}\frac{\delta G}{\delta q^*}\right)\,\mathrm{d}T. \tag{21}$$

Note that H does not appear here because F is conserved by NLS.

Now consider the two NLS conserved quantities

$$E = \int_0^{T_0} |q|^2\,\mathrm{d}T, \tag{22}$$

$$J = \frac{\mathrm{i}}{2}\int_0^{T_0}\left(q\partial_T q^* - q^*\partial_T q\right)\,\mathrm{d}T, \tag{23}$$

which constitute the energy E and the net energy flux J, respectively.

Even in the perturbed flow, the quantities E and J must be constant in Z on traveling wave solutions of the form (6) because the integrands in these quantities will be functions of the form $f(T-cZ)$, so that the dependence of these quantities on Z is integrated out. We may use this fact to obtain necessary (Melnikov) conditions for the persistence of traveling waves.

Suppose that a periodic traveling wave of the NLS equation with period T_0, represented in polar form by

$$q^0 = Q^0(\xi)\exp[-\mathrm{i}\alpha^0 Z + \mathrm{i}S^0(\xi)], \qquad \xi = T - c^0 Z, \tag{24}$$

persists as a periodic traveling wave with period T_0 of the perturbed NLS equation. This means there is a family of solutions of the perturbed equation given by

$$q^\epsilon = Q^\epsilon(\xi)\exp[-\mathrm{i}\alpha^\epsilon Z + \mathrm{i}S^\epsilon(\xi)], \qquad \xi = T - c^\epsilon Z, \tag{25}$$

for all $0 < \epsilon < \epsilon_0$, for some $\epsilon_0 > 0$, and such that

$$\lim_{\epsilon\to 0} Q^\epsilon(\xi) = Q^0(\xi), \quad \lim_{\epsilon\to 0} S^\epsilon(\xi) = S^0(\xi), \tag{26}$$

uniformly in ξ in the C^∞ topology of periodic functions on $[0, T_0]$, and

$$\lim_{\epsilon\to 0} \alpha^\epsilon = \alpha^0, \quad \lim_{\epsilon\to 0} c^\epsilon = c^0. \tag{27}$$

Because the E and J must be conserved quantities on these persisting traveling wave solutions, we have

$$\frac{\mathrm{d}E}{\mathrm{d}Z}(q^\epsilon) = \frac{\mathrm{d}J}{\mathrm{d}Z}(q^\epsilon) = 0. \tag{28}$$

From our assumptions it follows from continuity that

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\frac{\mathrm{d}E}{\mathrm{d}Z}(q^\epsilon) = \int_0^{T_0}\left(\frac{\delta E}{\delta q^*}\frac{\delta G}{\delta q} + \frac{\delta E}{\delta q}\frac{\delta G}{\delta q^*}\right)\bigg|_{(q^0)} \mathrm{d}T = 0, \tag{29}$$

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\frac{\mathrm{d}J}{\mathrm{d}Z}(q^\epsilon) = \int_0^{T_0}\left(\frac{\delta J}{\delta q^*}\frac{\delta G}{\delta q} + \frac{\delta J}{\delta q}\frac{\delta G}{\delta q^*}\right)\bigg|_{(q^0)} \mathrm{d}T = 0. \tag{30}$$

These are therefore necessary conditions for persistence. Note that we only need to know the unperturbed traveling waves to evaluate these conditions.

Using (20), (21), and (24), these conditions become

$$C_E = \int_0^{T_0} (D_g[Q_\xi^2 + Q^2 S_\xi^2] + g'(Q^2)Q^2)\,\mathrm{d}\xi + h'(I(q))\int_0^{T_0} Q^2\,\mathrm{d}\xi = 0, \tag{31}$$

$$C_J = \int_0^{T_0} (D_g[Q_\xi^2 - 2QQ_{\xi\xi} + Q^2 S_\xi^2] + g'(Q^2)Q^2) \times S_\xi\,\mathrm{d}\xi + h'(I(q))\int_0^{T_0} Q^2 S_\xi\,\mathrm{d}\xi = 0, \tag{32}$$

where we have set $Q = Q^0$ and $S = S^0$ for simplicity.

Integrating the term with integrand $-2QQ_{\xi\xi}$ in the second condition by parts once and using $S_\xi = \mu/Q^2 + c$ (from Eq. (8)), we obtain a useful alternate form of the conditions

$$C_E = \int_0^{T_0} f(Q, Q_\xi, S_\xi)\,\mathrm{d}\xi = 0, \tag{33}$$

$$C_J = \int_0^{T_0} f(Q, Q_\xi, S_\xi)\left(\frac{\mu}{Q^2} + c\right)\mathrm{d}\xi + 2D_g\int_0^{T_0}\left(cQ_\xi^2 - \mu\frac{Q_\xi^2}{Q^2}\right)\mathrm{d}\xi = 0, \tag{34}$$

where

$$f(Q, Q_\xi, S_\xi) = D_g[Q_\xi^2 + Q^2 S_\xi^2] + (g'(Q^2) + h'(I(q)))Q^2. \tag{35}$$

Recall that the NLS traveling wave solutions (6)–(11) are parameterized by the modulation number m, the phase winding number n, the period T_0, and two parameters κ and δ which lie in the simplex depicted in Fig. 1. When evaluated on these solutions at fixed values of m, n, and T_0, the selection conditions (33) and (34) form a 2×2 system of nonlinear equations

$$C_E(\kappa,\delta)=0, \quad C_J(\kappa,\delta)=0, \tag{36}$$

in the parameters κ and δ. Each condition determines some one-dimensional curves in the κ, δ simplex on which the condition is satisfied. We call all such curves "root-lines".

The conditions will be simultaneously satisfied only at points in the simplex where the root-lines from the two conditions intersect. Because the functions $C_E(\kappa,\delta)$ and $C_J(\kappa,\delta)$ are generically linearly independent of each other over the κ, δ simplex, this means that traveling waves will typically persist only at isolated points in the simplex. This is the first major insight we can draw from the selection conditions.

As proven in CLL2 for the case $h(\xi)=0$, a *sufficient* condition for an NLS traveling wave to persist as some traveling wave, with possibly different periods in the amplitude and phase, is that the root lines intersect transversely at the point representing the traveling wave. This condition is equivalent to requiring that the Jacobian determinant of the conditions C_E and C_J with respect to the parameters δ and κ is not zero. However, it is shown in CLL2 that if another transversality condition is met, which is difficult to evaluate but is met generically, then the wave will persist as a fully periodic wave with period T_0.

4. Summary of relevant results from CLL1–CLL3

We now briefly summarize some relevant results obtained in CLL3, which are derived using results of CLL1 and CLL2, upon which we will build in the next section for the specific example of the cubic CGL perturbation (3). We point out which results are more general than the cubic CGL case wherever they occur.

We remark that the results described in this section are qualitatively similar for the case of EDFAs, for which the CGL saturation term sz in Eq. (3) is replaced with

$$h'(I(q))=sI(q). \tag{37}$$

For example, the number of selection conditions and perturbation parameters remain the same, and the qualitative role of this term as a nonlinear saturation term that increases nonlinearly in magnitude with pulse energy remains unaltered. The primary effect of this replacement is therefore only to slightly shift the values of κ and δ of the persisting waves.

We first describe the persistence of the cw waves. This case is quite trivial, but is very important for understanding the rest of the Letter.

The cw wave solutions occur at $\kappa=0$, at which both selection conditions C_E and C_J reduce to the single condition

$$D_g\left(\frac{2\pi n}{T_0}\right)^2+g'(\eta^2\delta^2)+h'(\eta^2\delta^2)=0, \tag{38}$$

where $\eta=m\pi/T_0$. For the CGL perturbation (3) this condition yields selection of exactly one cw wave for each n and m which occurs at

$$\delta=\delta^*\equiv\frac{T_0}{m\pi}\sqrt{\frac{R-D_g(2\pi n/T_0)^2}{s}}, \quad \kappa=0. \tag{39}$$

Different values of m for the same n yield the same solution of Eq. (2) (recall that the cw waves are degenerate limits of traveling wave families).

For values of the parameters R, D_g, s, n, m, and T_0 such that $\delta^*<1$, it is shown in CLL3 for the cubic CGL perturbation that root-lines of C_E and C_J emanate tangentially from the selected cw wave at $(\delta,\kappa)=(\delta^*,0)$ into the simplex. This fact can be exploited to prove the existence of crossings in the simplex interior. These root-lines can be seen, for example, in Figs. 2 and 3. Note that the transversality (sufficiency) condition is *not* met at the selected cw wave. However, in this trivial case, direct calculation of the cubic CGL cw waves yields exactly Eq. (39) as the value for δ.

We now consider the persistence of uncentered linear phase traveling waves at $\delta=1$, $0<\kappa\leqslant 1$, which have the form (16). In this case, only solutions with $n=0$ have a possibility of persistence. To see this, assume that condition C_E (Eq. (33)) has been satisfied for one of the solutions (16) for some choice of n, m, T_0 and at some κ. Under this assumption, using the fact that the traveling wave parameter μ (given by Eq. (10)) is identically zero at $\delta=1$, the condition C_J (Eq. (34)) reduces to

$$C_J=2D_g c\int_0^{T_0} Q_\xi^2\,d\xi=0. \tag{40}$$

This immediately implies that c must also vanish, and therefore that only traveling waves at $\delta=1$ with $n=0$

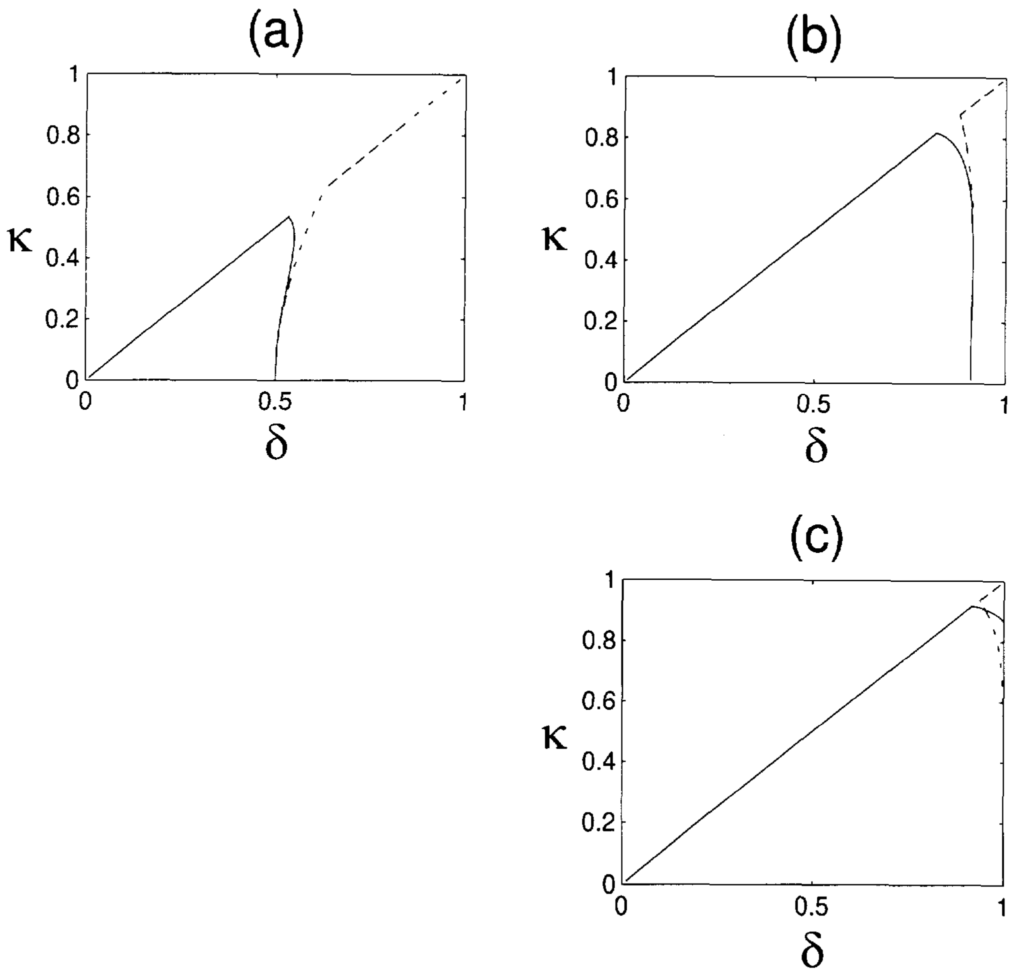

Fig. 2. Typical progression for $m \geqslant 2n$, at $T_0 = 1$, $m = 2$, $n = 1$, and $s = 1$. (a) $R = 59.22$: no crossing exists. (b) $R = 104.9$, crossing is just being born at the selected cw wave. (c) $R = 150$, persisting crossing can still be seen.

(because $c = 2\pi n/T_0$) have a possibility of persistence. Note that this argument is quite general and did not depend on the specific choice of the perturbation functions g and h.

We must now discuss whether or not any of the linear phase waves with $n = 0$ and $\delta = 1$ do in fact persist. These solutions have the simple form

$$q(Z,T) = \eta\, \mathrm{dn}(\eta[T-\theta_0], \kappa) \times \exp[i(\tfrac{1}{2}\eta^2(2-\kappa^2)Z + \sigma_0)]. \quad (41)$$

For these waves, using the facts that $\mu = c = S_\xi = 0$, both selection conditions are found to reduce to a single condition on κ (with other parameters T_0, m, and the perturbation g' fixed)

$$C(\kappa) = \int_0^{T_0} (D_g[\eta^2(2-\kappa^2)Q_0^2 - Q_0^4] + g'(Q_0^2)Q_0^2)\, d\xi + h'(I(Q_0)) \int_0^{T_0} Q_0^2\, d\xi = 0, \quad (42)$$

where $Q_0 = \eta\, \mathrm{dn}(\eta\xi, \kappa)$, $\eta = 2m K(\kappa)/T_0$.

In CLL3 it is shown for the cubic CGL perturbation (3) that $C(\kappa)$ has exactly one root if and only if R is greater than a certain critical value R_c. For a given choice of m, the root is born at $\kappa = 0$, $\delta = 1$, at

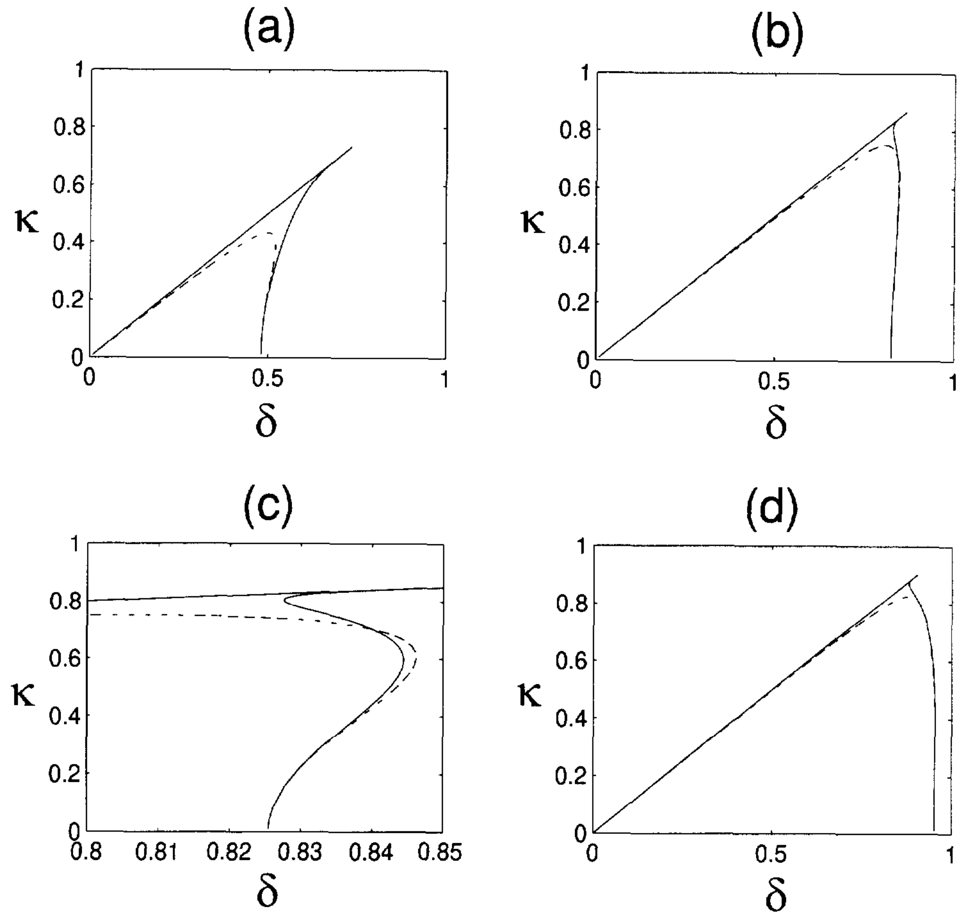

Fig. 3. Typical progression for $m < 2n$, at $T_0 = 1$, $m = 1$, $n = 1$, and $s = 4.5$. (a) $R = 60$, and no internal crossing is evident. (b) $R = 100$, the crossing exists, and can be seen more clearly in (c). (d) $R = 120$, and the crossing seen in (b) has disappeared.

$R = R_c$ (all other parameters fixed). At this point the selected cw wave, under an infinitesimal CGL perturbation ($0 < \epsilon \ll 1$), loses stability with respect to sideband perturbations with wavenumber k_m. Therefore, selection begins when a linear phase wave bifurcates from a cw wave in a hopf bifurcation.

We now summarize results from CLL3 for the persistence of general periodic traveling waves in the simplex. In CLL3 it is shown that nontrivial uncentered waves can only become selected via bifurcations from cw waves if $\mathrm{sign}(\mu)n > 0$. Thus, we consider here only the families for which this condition holds. For these families, the results divide into two subcases. First of all, for $m \geqslant 2n$, the following facts hold. As R is increased from zero, there is a critical value of R at which the selected cw wave, under an infinitesimal cubic CGL perturbation ($0 < \epsilon \ll 1$), loses stability with respect to sideband perturbations with wavenumber k_m. At this point a transverse crossing of the root-lines bifurcates from the cw wave into the simplex, indicating that a nontrivial periodic traveling wave becomes selected. This result is proven by expanding the root-lines emanating from the selected cw wave point in the parameter κ. This traveling wave remains selected as $R \to \infty$. Fig. 2 illustrates this progression with a typical example.

For the subcase case $m < 2n$, it is proven in CLL3 that under an infinitesimal cubic CGL perturbation, if s is sufficiently large, there is a critical value of R at which the selected cw wave gains stability with respect to sideband perturbations with wavenumber k_m. At this point a transverse crossing of the root-lines is introduced into the simplex, indicating that a nontrivial periodic traveling wave becomes selected. There is then a larger value of R at which the selected cw wave again loses stability with respect to sideband perturbations with wavenumber k_m, at which point a transverse crossing of the root-lines disappears from the simplex. Fig. 2 illustrates this second type of progression with a typical example.

Whether or not a persisting solution still exists at $\epsilon = 1$, and whether or not it is stable, are two important remaining questions. Here, we only make the following remarks.

It was proven for the cubic CGL perturbation by Takáč [15] that for any ϵ a periodic traveling wave with winding number n and modulation number m bifurcates from a cw wave (also with winding number n) when the cw wave loses stability with respect to wavenumber k_m sideband perturbations. This result establishes the existence for arbitrary values of ϵ and an infinitesimal range in R near the bifurcation point. The approach used here exactly complements Takáč's result by allowing existence to be established for an arbitrary range in R and an infinitesimal range in ϵ.

Secondly, we remark that numerical simulations by various authors [14,13] demonstrate that there are regimes in the parameters T_0, R, s, and ϵ for which the persisting linear phase solution with $n = 0$ is stable.

5. Persistence in the autosoliton limit

We now explore the persistence conditions in the autosoliton limit. As $T_0 \to \infty$, the only subset of solutions described by (6)–(11) that have nonvanishing amplitude, and therefore the only solutions that have a chance of satisfying the balances required for persistence by the selection conditions, are those in the soliton limit. As described in Section 2, these have the asymptotic form (17), where the limit $T_0 \to \infty$ is taken while letting κ and δ tend to 1 along specific paths in the simplex, in such a way that the amplitude η and speed c remain constant.

When the conditions C_E and C_J are evaluated on the asymptotic solutions (17) for the CGL perturbation, we obtain

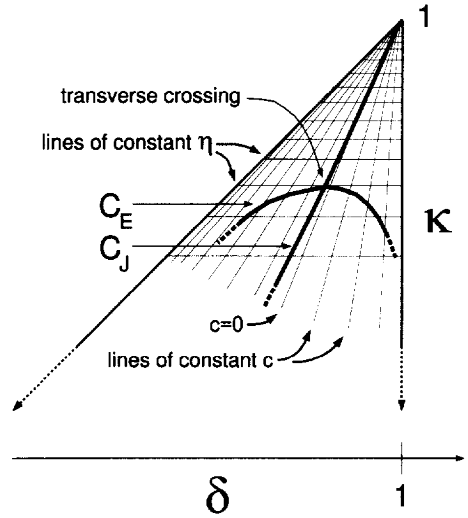

Fig. 4. Upper right-hand corner of the κ, δ simplex, illustrating the transverse crossing in the infinite period limit.

$$C_E = \eta^2 + \frac{c^2}{D_g/3 + 2s/3} - \frac{R}{D_g/3 + 2s/3} + \mathrm{O}(1/T_0) = 0, \tag{43}$$

$$C_J = cC_E + 4cD_g\eta + \mathrm{O}(1/T_0) = 0, \tag{44}$$

for the selection conditions in the simplex. These conditions are functions of δ, κ, and T_0 because c and η are unique functions of δ, κ, and T_0 for a given choice of m and n.

We first address the case taken strictly in the limit: in this limit, the chain of solitons (17) reduces to a single one-soliton solution,

$$q(Z,T) = \eta\,\mathrm{sech}(\eta[T - cZ]) \times \exp[icT + \tfrac{1}{2}i(\eta^2 - c^2)Z], \tag{45}$$

where η and c are the two principal soliton parameters (we suppress the two trivial phase parameters).

From Eq. (43), it can be immediately seen that the one soliton solutions that satisfy C_E occur on an ellipse in the parameters η and c determined by the choice of R, D_g and s. Clearly, of all the choices of η and c for which C_E is satisfied, the only one which also satisfies C_J occurs at $c = 0$. This shows that of all the one soliton solutions, only the solution (up to trivial phase constants),

$$q = \eta\,\mathrm{sech}(\eta T)\,\mathrm{e}^{i\eta^2 Z/2}, \tag{46}$$

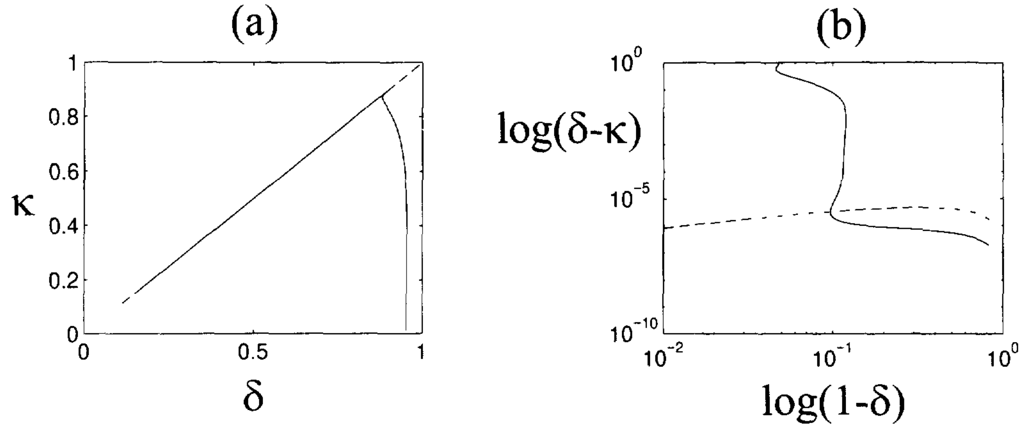

Fig. 5. Same parameters as Fig. 3a, but this time showing a different root-line of C_J. In (a), only the end segments of the dashed C_J root-line, which lies along the simplex hypotenuse, can be seen. (b) Same picture in alternative coordinates. In these coordinates, both lines and the crossing can be clearly seen.

can persist, where

$$\eta = \frac{R}{D_g/3 + 2s/3}. \tag{47}$$

Sufficiency in this case can also be shown. It was shown previously by Pereira and Stenflo [7] that the cubic CGL equation has an exact solution of the form

$$q = K\,\mathrm{sech}(LT)^{1+iM}\,e^{iNZ}, \tag{48}$$

where K, L, M, and N are algebraic functions of ϵ, R, D_g and s (we omit these expressions). When the limit of these expressions as $\epsilon \to 0$ is taken, we find that the solitary wave (48) reduces exactly to the persisting soliton (46) in the limit. This proves the persistence for this one stationary soliton, and the Melnikov analysis proves the nonpersistence of all other one-soliton solutions.

We remark that if the perturbation is of cubic CGL form except with a saturation term that depends on the average intensity $I(q)$ instead of $|q|^2$, as in (37), then in the autosoliton limit the saturation effect is absent (recall that $I(q) \to 0$ as $T_0 \to \infty$). However, the results above with the Pereira and Stenflo [7] solution (48) still go through in this case, yielding Eq. (47) with s set to zero.

We now explore the relationship of the persisting periodic waves of large period to the persisting one-soliton (46). Emanating from the corner of the simplex are lines of constant speed (c), which range from $c = 2\pi n/T_0$ at $\delta = 1$ to $c = -\infty$ on the hypotenuse of the simplex. Crossing these lines transversely are horizontal lines of constant η, with values ranging from $\eta = m\pi/T_0$ at $\kappa = 0$, $0 \leqslant \delta \leqslant 1$, to $\eta = \infty$ at $\delta = \kappa = 1$. Fig. 4 illustrates the coordinate system created by these lines, and also the following observation. Expression (44) for C_J implies that C_J will have a root-line asymptotic to the $c = 0$ path in the right-hand corner of the simplex, and will have no other root-lines nearby points at which C_E is satisfied. If the period T_0 or if the gain R is large enough, then expression (43) for C_E, because it is the equation for an ellipse in c and η, implies that a root-line for C_E will cross the root-line for C_J transversely, as illustrated in Fig. 4. In the limit $T_0 \to \infty$, this crossing becomes exactly orthogonal in the coordinates c and η, ensuring satisfaction of the transversality condition.

Thus we see that every family of NLS traveling waves specified by a choice of m and n, in the limit $T_0 \to \infty$, has a persisting solution with the leading order form

$$q(Z,T) = \sum_{j=-\infty}^{j=+\infty} \eta\,\mathrm{sech}\left(\eta\left[T - j\frac{T_0}{m} - \theta_0\right]\right) \times \exp\left(\frac{i}{2}\eta^2 Z + \frac{2\pi n}{m} j + i\sigma_0\right), \tag{49}$$

where η is given by (47). These persisting solutions are the periodic analogues of (46). Note that these

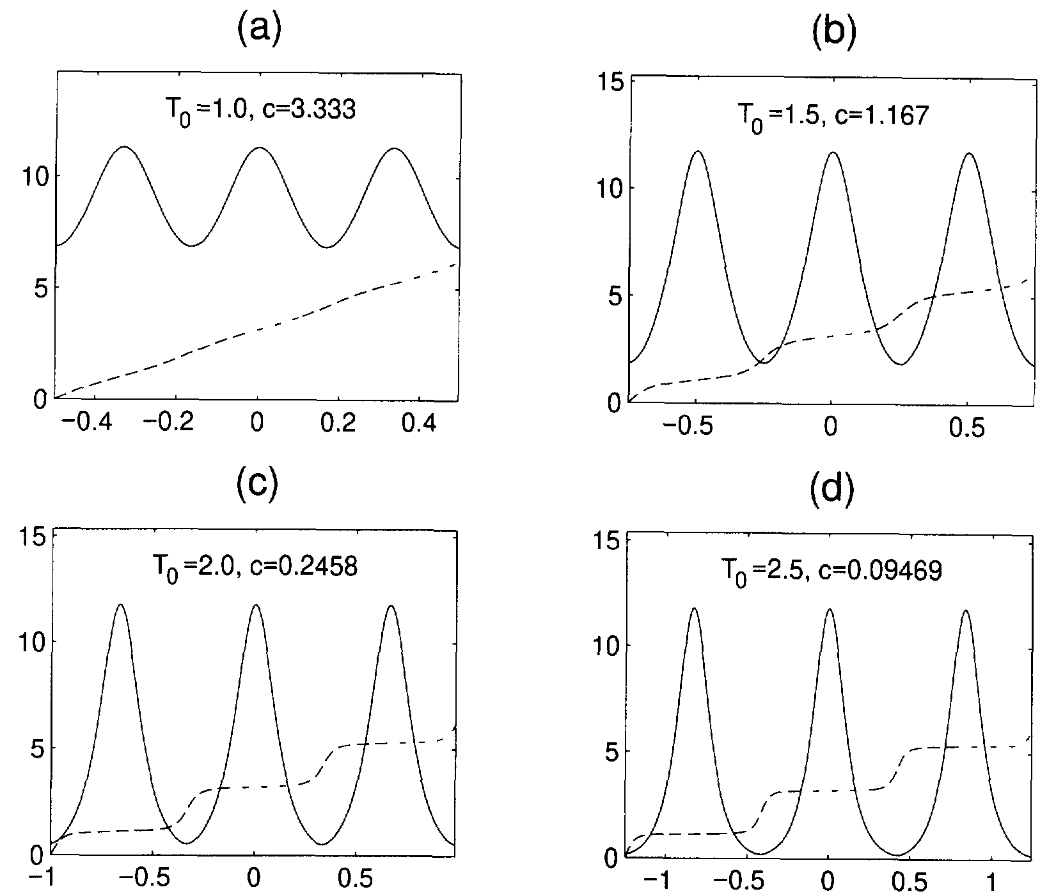

Fig. 6. Selected pulses for $m = 3$, $n = 1$, $R = 40$, and $s = 1$, at (a) $T_0 = 1$, (b) $T_0 = 1.5$, (c) $T_0 = 2.0$, (d) $T_0 = 2.5$. The phase profiles over the duration of the individual pulses are seen to flatten out and the speed c goes to zero as T_0 increases, in accordance with predictions.

pulses are stationary to leading order in $1/T_0$.

The results from CLL2–CLL3 quoted in the last section show that periodic wavetrains with nonzero speed can persist in the cubic CGL equation. The result above shows that if a family of these persisting wavetrains has members with periods tending to infinity, or at values of R tending to infinity, then the speed of these persisting solutions in the infinite period limit must tend to zero, and the amplitudes and pulse widths must tend to the value of η given by (47).

However, we have seen in the previous section that some families of persisting solutions, in particular the families discussed there with $m < 2n$ related to bifurcations of cw waves, do not continue at large R (likewise as $T_0 \to \infty$). But to the contrary, the results here imply that for $m < 2n$ (as well as $m \geqslant 2n$), there are persisting solutions. The resolution of this apparent paradox lies in the fact that for $m < 2n$, the condition C_J actually has more than one root-line in the simplex, and one of these lines is asymptotic to the $c = 0$ line in the upper right-hand corner of the simplex and gives persisting solutions in the infinite period limit, while the other line gives the families discussed in the previous section. Fig. 5a and Fig. 5b show another look at the example for the parameters corresponding to Fig. 3c, at $m = 1$, $n = 1$. Here, the other root-line of C_J has been located which lies very close to the hypotenuse of the simplex. Both figures show the same root-line, but Fig. 5b uses logarithmic coordinates which clearly show the nearly orthogonal crossing predicted by the analysis. Thus we see that there are particular families which are related to persisting solitary pulses in the limit, and some which are not.

Fig. 6 shows an example illustrating the basic results of this section. The example shown in this figure occurs at $m = 3$, $n = 1$, with parameters chosen so that there is a crossing fairly close to the upper right-hand corner of the simplex at $T_0 = 1$. Then, the series of selected NLS wavetrains for $m = 3$, $n = 1$, which occur as T_0 is increased from $T_0 = 1$, are shown in Figs. 6a–6d. Note that, as predicted by the analysis, the phase across each pulses flattens out, and the speed tends quickly to zero. Also note that the amplitude and pulse width change little as soon as the pulses become somewhat separated.

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