



## Approximate evolution of lump initial conditions for the Benjamin–Ono equation

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### Abstract

In this paper it is shown how the approximate evolution of soliton forming initial conditions for the Benjamin–Ono equation can be described completely in terms of an integrable two-dimensional phase-plane system. It is found that there is good agreement between these approximate solutions and full numerical solutions of the Benjamin–Ono equation, both in the temporal evolution of the soliton amplitude and in the final soliton state for a wide range of initial conditions. ©1998 Elsevier Science B.V. All rights reserved.

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### 1. Introduction

In this work the approximate method of Kath and Smyth [1] will be used to study the evolution of soliton forming initial conditions for the Benjamin–Ono (BO) equation. This approximate method was developed in the context of the Korteweg–de Vries equation and includes the effect on the developing soliton of the dispersive radiation shed as the soliton evolves. The method of Kath and Smyth can be used to determine solitary wave evolution for equations for which there are no inverse scattering solutions, as was done by Smyth and Worthy [2] for the mKdV equation and as in the present case. The BO equation contains a non-local term and the method of Kath and Smyth will be adapted to account for it. It is found that the derived approximate equations, which incorporate the effect of radiation on the evolving soliton, can be integrated exactly. In all other applications of the method of Kath and Smyth, the approximate (ordinary differential) equations describing the evolution of the soliton have had to be solved numerically [1,2]. This explicit solution of the approximate equations allows a detailed classification of all possible evolution paths of a special class of initial conditions. It also allows the global stability of the critical points of the approximate equations to be determined. In previous applications of the approximate method to the Korteweg–de Vries and mKdV equations [1,2], a complete stability analysis of the solutions of the approximate equations was not possible, but it was believed, and confirmed numerically, that the critical points were globally stable. The phase-plane analysis presented here confirms this in the case of the BO equation. The solutions obtained from the approximate equations are compared

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with full numerical solutions of the BO equation obtained using the pseudo-spectral method of Fornberg and Whitham [3].

**2. Formulation**

Let us consider the BO equation in the normalisation

$$\frac{\partial u}{\partial t} + 2u \frac{\partial u}{\partial x} + \frac{\partial^2}{\partial x^2} (H(u)(x)) = 0, \tag{1}$$

where

$$H(u)(x) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{u(y)}{y-x} dy. \tag{2}$$

Let us also take the initial condition

$$u(x, 0) = \frac{AB^2}{x^2 + B^2}. \tag{3}$$

This initial condition is a soliton solution of the BO equation when  $B = 2/A$ . Fig. 1 shows a numerical solution of the BO equation for this initial condition with  $A = 1.5$  and  $B = 1$ . It can be seen that the initial condition evolves into a soliton followed by a growing (in extent) shelf leading a dispersive wavetrain.

To study the evolution of the initial condition (3) we use the mass and momentum conservation laws for the BO equation, which are

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (u^2 + H_x(u)) = 0, \tag{4}$$

$$\frac{\partial}{\partial t} \left( \frac{1}{2} u^2 \right) + \frac{\partial}{\partial x} \left( \frac{2}{3} u^3 + u[H(u)]_x \right) - u_x[H(u)]_x = 0. \tag{5}$$

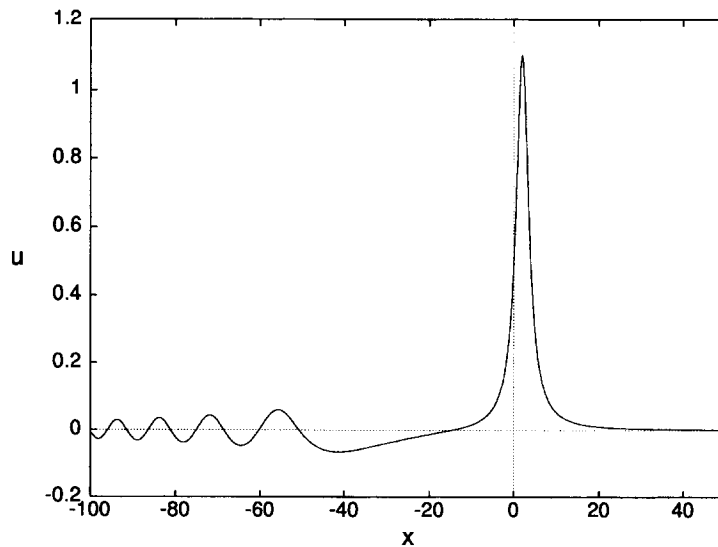


Fig. 1. Numerical solution of the BO equation (1) at  $t = 50$  for initial condition (3) with  $A = 1.5$  and  $B = 1.0$ .

Following Kath and Smyth [1] and Smyth and Worthy [2], we use the trial function

$$u(x, t) = u_0(x, t) + u_1(x, t) \tag{6}$$

with

$$u_0 = \frac{ab^2}{\theta^2 + b^2}, \quad \theta = x - \zeta(t) \tag{7}$$

in the conservation laws (4) and (5) to find approximate equations for the evolution of the soliton. The parameters  $a$ ,  $b$  and  $\zeta$  are functions of  $t$ . The function  $u_1$  accounts for the dispersive radiation behind the evolving soliton. We note from Fig. 1 that the radiation has smaller amplitude than the soliton, so that  $|u_1| \ll u_0$ . From the initial condition (3) it can be seen that  $a(0) = A$  and  $b(0) = B$ . Furthermore, when  $b = 2/a$ ,  $\dot{\zeta} = V = a/2$  and  $u_0$  is the soliton solution of the BO equation so that  $u_1 = 0$ .

### 3. Approximate equations

Linearising the BO equation (1) it can be found that the dispersion relation for linear waves is  $\omega = -k|k|$ , where  $k$  is the wave number and  $\omega$  is the frequency. Hence linear dispersive waves have negative group and phase velocities and lie behind the evolving pulse. This can also be seen from the numerical solution shown in Fig. 1. Therefore, in the region  $x \geq \zeta(t)$  there is no dispersive radiation, so that  $u_1 = 0$  there. Integrating the mass conservation equation (4) in the region  $x \geq \zeta(t)$  gives the exact expression

$$\frac{d}{dt} \int_{\zeta}^{\infty} u \, dx = \int_{\zeta}^{\infty} u_t \, dx - \dot{\zeta} u(\zeta(t), t) = -[u^2 + H(u)]_{\zeta}^{\infty}. \tag{8}$$

Since  $u_1 = 0$  for  $x \geq \zeta(t)$ , substituting the trial function (6) into the mass conservation equation (8) gives

$$\pi \frac{d}{dt} (ab) = 2a^2 - 2ab^{-1} - 2aV, \tag{9}$$

where  $V = \dot{\zeta}$ . This is one equation for the three unknowns  $a$ ,  $b$  and  $V$ .

As in Kath and Smyth [1] and Smyth and Worthy [2] the equation for the soliton velocity  $V$  is obtained from the moment of momentum equation

$$\frac{d}{dt} \int_{-\infty}^{\infty} \frac{1}{2} x u^2 \, dx + \frac{2}{3} \int_{-\infty}^{\infty} (u^3)_x \, dx + \int_{-\infty}^{\infty} x u [H(u)]_{xx} \, dx = 0. \tag{10}$$

Taking this moment equation to first order, so that  $u = u_0$  with  $u_0$  given by (7), we obtain

$$V = a - \frac{1}{b}, \tag{11}$$

which reduces to the soliton speed when  $ab = 2$ . As in [1,2] higher-order corrections to this velocity become important and they are accounted for by taking the same functional dependence as (11), but with a different weight. In doing this, the appropriate limiting behaviour for a soliton must be maintained. We therefore take

$$V = 2(1 + \mu)a - \frac{3 + 4\mu}{b}, \tag{12}$$

where the parameter  $\mu$  will be determined later. It is noted that when  $\mu = -0.5$  the velocity (11) is obtained and that the correct soliton velocity  $V = a/2$  is obtained when  $a$  and  $b$  are related as for a soliton ( $ab = 2$ ), independent

of  $\mu$ . The parameter  $\mu$  acts as a decay constant which determines how fast the solution evolves to the steady state, but does not affect the final steady state to any great degree.

We shall now derive the last equation for the three unknowns  $a$ ,  $b$  and  $V$ . This equation is determined from the equation for momentum conservation (5) and by examining the mechanism for mass and momentum loss from the soliton. We note from Fig. 1 that there is a shelf behind the evolving soliton. To determine the effect of the radiation in the mass conservation equation, let us integrate the mass conservation (4) from the trailing edge of the shelf, say at  $x = L(t)$ , to  $x = \infty$ . We then obtain

$$\frac{d}{dt} \int_L^{\infty} u \, dx = -\dot{L}u(L, t) - \int_L^{\infty} (u_x^2 + [uH(u)]_{xx}) \, dx. \quad (13)$$

Assuming that the shelf is flat and moves with the soliton velocity  $V$ , we obtain

$$\frac{d}{dt} \int_L^{\infty} u \, dx = -Vu_{\infty}, \quad (14)$$

where  $u_{\infty}$  is the height of the shelf, which at this stage is unknown. However, on integrating the mass conservation equation (4) for  $-\infty < x < \infty$ , we have the total conservation of mass equation

$$0 = \frac{d}{dt} \int_{-\infty}^{\infty} (u_0 + u_1) \, dx = \frac{d}{dt} (\pi ab) + \frac{dM}{dt}, \quad (15)$$

where

$$M = \int_{-\infty}^{\infty} u_1 \, dx \quad (16)$$

is the mass in the radiation. Since the dispersive radiation lies behind the evolving soliton in  $x < \zeta(t)$ , we see from comparing the mass conservation results (14) and (15) that

$$\frac{dM}{dt} = Vu_{\infty}. \quad (17)$$

This expression just states that the rate of change of mass in the radiation is the flux of radiation shed from the shelf, and hence from the evolving soliton.

To close the system of equations, we now consider conservation of momentum. Integrating the momentum conservation equation (5) from the edge of the shelf  $x = L$  to  $x = \infty$ , we obtain

$$\frac{d}{dt} \int_L^{\infty} \frac{1}{2} u^2 \, dx = -\frac{1}{2} \dot{L}u^2(L, t) + \frac{2}{3} u^3(L, t) + u(L, t)[H(u)]_x(L, t) - \int_L^{\infty} u_x [H(u)]_x \, dx. \quad (18)$$

Now  $[H(u)]_x = H[u_x]$  and

$$\int_{-\infty}^{\infty} u_x H[u_x] \, dx = 0. \quad (19)$$

Hence to  $O(u_\infty^3)$  the momentum conservation equation (18) becomes

$$\frac{d}{dt} \int_L^\infty \frac{1}{2} u^2 dx = -\frac{1}{2} \dot{L} u^2(L, t). \tag{20}$$

Now  $u^2 = u_0^2 + 2u_0u_1 + u_1^2$ . Therefore, since the radiation lies behind the evolving soliton the cross term  $u_0u_1$  is zero. We may then take  $u = u_0$  to  $O(u_1^2)$  to obtain

$$\frac{d}{dt} \left( \frac{\pi}{2} a^2 b \right) = -\frac{V}{2} u_\infty^2 = -\frac{1}{2V} \left( \frac{dM}{dt} \right)^2 \tag{21}$$

on using (17), since  $u_\infty$  is assumed to be small.

In summary, the system of equations governing the evolving soliton is

$$\begin{aligned} \frac{d}{dt} (\pi ab) &= 2a^2 - 2ab^{-1} - 2aV, \\ \frac{d}{dt} (\pi ab) + \frac{dM}{dt} &= 0, \\ \frac{d}{dt} (a^2b) + \frac{8}{\pi a} \left( \frac{dM}{dt} \right)^2 &= 0, \\ V &= 2(1 + \mu)a - \frac{3 + 4\mu}{b}. \end{aligned} \tag{22}$$

In all other applications of the approximate method [1,2] the approximate equations governing the evolving pulse were solved numerically. What is remarkable for the present case of the BO equation is that these equations decouple and there exists an integral of motion which allows a complete description of the solutions of the approximate equations. Moreover, this integral of motion displays the line of critical points which are all the possible final soliton states.

To obtain this reduced system we note that by taking a new variable  $y = ab$  the equation for  $dM/dt$  decouples. The system can therefore be reduced to two equations

$$\begin{aligned} \frac{dy}{dt} &= \frac{1 + 2\mu}{\pi y} a^2(4 - 2y), \\ \frac{da}{dt} &= -\frac{1 + 2\mu}{\pi y^3} a^3(4 - 2y)[y + 8(1 + 2\mu)(2 - y)]. \end{aligned} \tag{23}$$

From this system it is clear that the line of critical points  $y = 2$  is the line of soliton states  $a = 2/b$ . Furthermore, this system has the integral of motion

$$F(a, y) = y^{7+16\mu} a^{-1} e^{16(1+2\mu)y^{-1}}. \tag{24}$$

Finally, it may be shown that the line of critical points is stable if  $\mu > -1/2$ .

The phase plane for the system (23) is shown in Fig. 2. It can be seen that the soliton can either gain or lose mass as it evolves and that the trajectories always evolve to the line of fixed points  $y = ab = 2$ , corresponding to a soliton solution of the BO equation. It also follows from the integral of motion (24) that for  $\mu \neq -7/16$  the minimum of

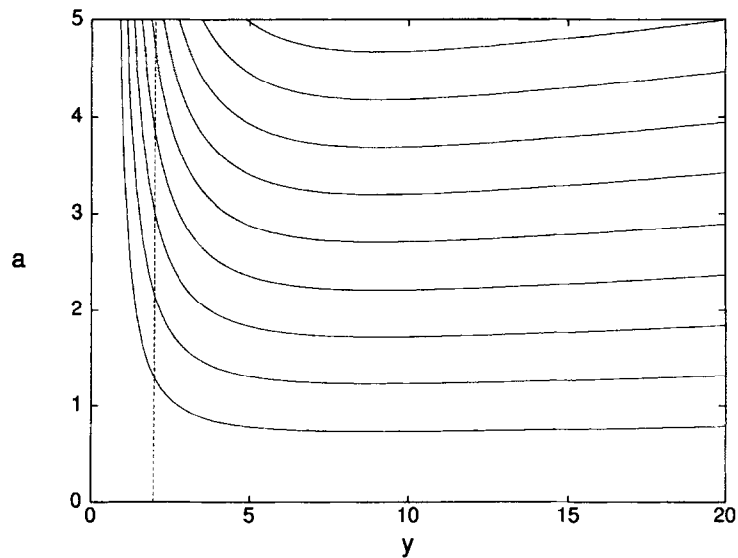


Fig. 2. Phase-plane for the system of equations (23). The vertical dashed line is the line of equilibrium points  $y = 2$ .

the trajectories is always to the right of the line of solitons  $y = 2$  and that the minimum of the trajectories lies on the line

$$y_{\min} = \frac{16(1 + 2\mu)}{7 + 16\mu}. \quad (25)$$

Thus the amplitude always decreases and the pulse gains mass if the initial condition has  $y$  less than the soliton value 2. This is because the pulse needs to gain mass to reach a soliton. On the other hand, if the initial condition has  $y$  greater than 2, the pulse must lose mass to reach a soliton, as seen in the phase plane of Fig. 2. The evolution of the amplitude for the initial value of  $y$  greater than 2 is more complicated however. If the initial  $y$  is less than  $y_{\min}$  and greater than 2 then the amplitude of the pulse always increases to the soliton solution. However, if the initial  $y$  is greater than  $y_{\min}$ , the amplitude first decreases until  $y$  reaches  $y_{\min}$ , then increases to the soliton. This behaviour of the amplitude for the initial  $y \leq y_{\min}$  conforms with full numerical solutions of the BO equation. However, the full numerical solutions do not show an initial decrease in amplitude for  $y > y_{\min}$ . In the next section  $\mu$  is found to have the value  $-0.42$ , so that  $y_{\min}$  is approximately 9. For such a large initial value of the mass of the pulse, it is not expected that the approximate equations will give good agreement with numerical solutions. Indeed for  $y > y_{\min}$ , 5 or more solitons eventually form out of the initial pulse.

#### 4. Comparison with numerical solutions

In this section solutions of the approximate equations (23) will be compared with full numerical solutions of the BO equation (1) obtained using the pseudo-spectral method of Fornberg and Whitham [3]. Before solutions of the approximate equations can be obtained, the parameter  $\mu$  must be determined. To determine this parameter, the full numerical solution of the BO equation (1) was calculated for the initial condition (3) with  $A = 1.5$  and  $B = 1.0$ . Then  $\mu$  was determined so as to provide the best comparison between the full numerical solution and the solution of the approximate equations (23). This gave  $\mu = -0.42$ .

The amplitude  $a$  of the pulse as a function of time  $t$  for the initial conditions  $A = 1.5$  and  $B = 1.0$  as given by the full numerical solution and by the solution of the approximate equations is shown in Fig. 3(a). It can be seen

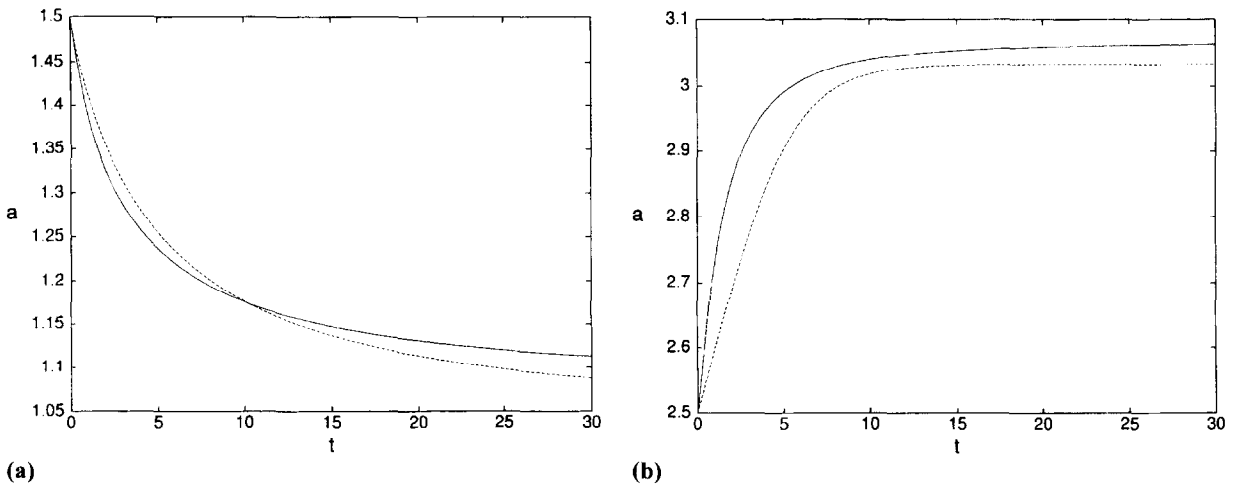


Fig. 3. Pulse amplitude as  $a$  as a function of time  $t$ . Full numerical solution (—); solution of approximate equations (---). (a) Initial condition  $A = 1.5$  and  $B = 1.0$ ; (b) initial condition  $A = 2.5$  and  $B = 1.0$ .

that the agreement between the two solutions is good, as is expected since the parameter  $\mu$  was chosen to give good agreement for this initial condition. Fig. 3(b) shows the same comparison as Fig. 3(a) for the initial condition  $A = 2.5$  and  $B = 1.0$ . It can be seen that the comparison is again very good and that the choice of  $\mu = -0.42$  is not just specific to the initial condition  $A = 1.5$  and  $B = 1.0$ .

The initial condition (3) evolves to a steady state soliton, for which  $ab = 2.0$ . Fig. 4 shows the steady state soliton amplitude  $a$  as a function of the initial pulse amplitude  $A$  for  $B = 1.0$ . Shown in the figure are the steady state amplitudes as given by the full numerical solution of the BO equation and as given by the approximate equations (23). It can be seen that there is very good agreement between the approximate and numerical solutions.

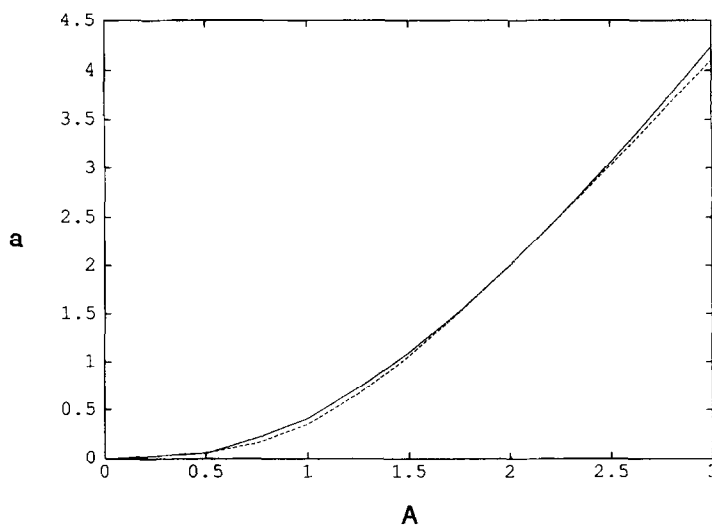


Fig. 4. Final soliton amplitude  $a$  as a function of initial amplitude  $A$ . Numerical solution (—); approximate solution (---). The initial pulse width is  $B = 1.0$ .

## 5. Conclusions

It has been established that a complete phase-plane picture for the evolution of special lump initial conditions for the BO equation can be obtained. It is remarkable that the system of approximate equations is integrable. The integral of motion allows the stability of the degenerate line of critical points which represents the final soliton steady states to be determined. This complete description is not available for previous equations studied using the approximate method used in the present work.

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