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Effect of low momentum quantum fluctuations on a coherent field structure

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In the present work the evolution of a coherent field structure of the sine-Gordon equation under quantum fluctuations is studied. The basic equations are derived from the coherent state approximation to the functional Schrödinger equation for the field. These equations are solved asymptotically and numerically for three physical situations. The first is the study of the nonlinear mechanism responsible for the quantum stability of the soliton in the presence of low momentum fluctuations. The second considers the scattering of a wave by the soliton. Finally the third problem considered is the collision of solitons and the stability of a breather. It is shown that the complete integrability of the sine-Gordon equation precludes fusion and splitting processes in this simplified model. The approximate results obtained are non-perturbative in nature, and are valid for the full nonlinear interaction in the limit of low momentum fluctuations. It is also found that these approximate results are in good agreement with full numerical solutions of the governing equations. This suggests that a similar approach could be used for the baby Skyrme model, which is not completely integrable. In this case the higher space dimensionality and the internal degrees of freedom which prevent the integrability will be responsible for fusion and splitting processes. This work provides a starting point in the numerical solution of the full quantum problem of the interaction of the field with a fluctuation.

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I. INTRODUCTION

In the past few years it has been shown that the Skyrme model [1] is related to the low energy limit of QCD [2]. This fact, together with progress in the approximate and numerical solution of strongly nonlinear equations, has renewed interest in a detailed study of the quantum mechanics of the Skyrme model. These studies are directed along two main lines. The first simplifies the model at the classical level to the so-called baby Skyrmion model, which in turn is related to the sine-Gordon equation [1,3,4]. In this simplified classical model extensive numerical studies have led to an increased understanding of the stability, scattering and interaction of coherent structures with radiation [3,4]. On the other hand, the second line has focused on the quantum effects of the full Skyrme model [5,6]. In these studies the quantization was either obtained by linearization around a field configuration [5] or by a finite dimensional approximation to the full problem [6].

Here we take a complementary approach. We study the $1+1$ sine-Gordon equation keeping all the degrees of freedom and quantize along the lines of [7–10]. This leads to a field equation which is strongly coupled with the equations for the fluctuations. In [10] the formalism for the functional coherent state approximation was fully developed and the possible advantages and disadvantages of various approaches designed for numerical purposes were discussed in detail. In particular, the closed-time path method with the Hartree factorization (see e.g. [11]) was applied in [10] to the static sine-Gordon field, and the approximate results obtained for phase transitions and stability were found to compare favorably with known exact results. In this same work static results were also obtained for more realistic field equations. Our present work differs from the above cited papers in that we choose to approximate the Green function (the variance kernel for the Gaussian ansatz) by a suitable parametrized trial function. This choice leads to a great simplification of the problem for the case of low momentum fluctuations. Thus, for example, the infinite system of partial differential equations of [7] become ordinary differential equations and, even though the field and the fluctuations in our formalism are strongly coupled, we are able to use some of the solutions of the classical sine-Gordon equation (since in $1+1$ dimensions it is completely integrable) to construct approximate solutions to the quantum problem, including the effect of the radiation generated by the quantum fluctuations. We then solve the equations numerically and these numerical solutions are compared with asymptotic solutions. Note also that the approach used in the present work is completely different to that of [12,13] where the wave functionals are constructed using Gaussian approximations to the functional Schrödinger equation for the sine-Gordon field. However, the particles are considered as higher excited states (in function space) of the linearized field equations. In our treatment the field equations are nonlinear and dynamic and different particles are represented by different nonlinear field configurations, not by higher order Hermite functionals as in [12,13].

Finally, it must be stressed that the approximate analytic results obtained here are valid in a strongly nonlinear regime and, in principle, do not depend on the $1+1$ nature of the model and could be used to study problems related to more realistic simplifications of the Skyrme model.

The paper is organized as follows. In Sec. II the detailed formulation of the quantum problem is stated, with the free parameters adjusted to mimic mesons and baryons. Section III is devoted to the study of the coherent state approximation and the derivation of the quantum equations for the field for low momentum fluctuations. In Sec. IV three problems are considered. The first is the nonlinear stability of a single soliton under the influence of quantum fluctuations. This stability is studied both numerically and asymptotically. In particular the asymptotic solution includes the damping effect of the radiation shed by the soliton due to the fluctuations. This asymptotic solution explains in detail the mechanism for the nonlinear stability of the soliton. The second problem studied is the scattering of a meson (wave) by a static soliton. The numerical solution for this problem shows that in this process the soliton is also stable. Finally the third problem studied concerns the collision of solitons and the quantum evolution of a bound state (soliton and an anti-soliton). It is shown that the complete integrability of the sine-Gordon simplification of the full Skyrme model to just one internal degree of freedom for the field precludes the processes of fusion and splitting. The processes of fusion and splitting are produced by the influence of all the internal degrees of freedom.

II. FORMULATION OF THE PROBLEM

For the basic structure we take the baby Skyrme model, which is a reduction of the full model with only two fields present [1,3]. This model, in turn, reduces to the sine-Gordon equation which is known to be completely integrable [3]. In these variables the Hamiltonian takes the form

$$H=mc^2 l\int\left[\frac{1}{2}\pi^2+\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2+l^{-2}(1-\cos\varphi)\right]dx, \tag{2.1}$$

where φ is an angle variable whose momentum π is given by

$$\pi=\frac{1}{c}\frac{\partial\varphi}{\partial t}, \tag{2.2}$$

m is the mass of the particle and l is a typical particle size. Using dimensionless variables $\tilde{x}=x/l$ and $\tilde{t}=ct/l$, we obtain, after dropping the tildes,

$$H=mc^2\int\left[\frac{1}{2}\pi^2+\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2+(1-\cos\varphi)\right]dx, \tag{2.3}$$

with $\pi=\partial\varphi/\partial t$. To mimic a baryon by means of the sine-Gordon soliton, we take $l\sim10^{-13}$ cm and $m\sim10^{-27}$ Kg, which gives an internal time of 10^{-23} sec.

The equation of motion derived from the Hamiltonian variational principle

$$\delta\int_{t_0}^{t_1}(\pi\dot{\varphi}-H)\ dt \tag{2.4}$$

is the sine-Gordon equation

$$\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x^2}+\sin\varphi=0. \tag{2.5}$$

The one dimensional Skyrmion is the soliton solution

$$\varphi=-4\arctan[\exp(-(x-vt)/\sqrt{1-v^2})], \tag{2.6}$$

of Eq. (2.5), which represents a localized deformation at $x=vt$. In this context v must be taken to satisfy $v\ll1$, since the Skyrme model is only consistent for small energies. The linear travelling periodic wave solutions of the sine-Gordon equation are interpreted as pions. Notice that the nonlinear model contains linear waves which describe bosons and nonlinear structures which describe fermions. In his original work Skyrme suggested that the fermionic part of the Lagrangian is needed only just to count the number of localized states of finite amplitude of the bosonic field [1]. In this interpretation, fermions are just a point approximation to nonlinear localized bose fields. It has also been suggested on rather general grounds that the canonical quantization of fields (as bosons) gives Fermi-Dirac type statistics for the kinks [15]. In the sine-Gordon model the exclusion principle holds for kinks, since we know that for the general exact solution there is no solution with two identical solitons [14].

In this article, we shall consider a canonical quantization using the functional Schrödinger picture. It is to be noted that all quantizations for the Skyrme model cited in the Introduction make the same assumption. However, in this work we differ from previous treatments in that we shall keep infinitely many degrees of freedom in the classical field and reduce the dimensionality of the space of fluctuations. The final result will be shown to be a system which consists of a partial differential equation (similar to the sine-Gordon equation) for the field coupled to a system of nonlinear ordinary differential equations which control the fluctuations.

In the field configuration representation the Schrödinger equation takes the form

$$i\hbar\frac{\partial\Psi(\varphi)}{\partial t}=\hat{H}\Psi(\varphi), \tag{2.7}$$

where $\Psi(\varphi(x),t)\coloneqq\langle\varphi(x)|\Psi(t)\rangle$ is the amplitude for finding the field system characterized by the state vector $|\Psi(t)\rangle$ in the field configuration $\varphi(x)$ at time t. In this configuration representation the scalar product of two state vectors is given by the functional integration

$$\langle\Psi_1|\Psi_2\rangle=\int\Psi_1^*(\varphi,t)\Psi_2(\varphi,t)\mathcal{D}\varphi, \tag{2.8}$$

and the field operators are represented by functional kernels. Thus the field operator $\hat{\phi}(x)$ is represented by $\langle\varphi(x)|\hat{\phi}(x)|\Psi(t)\rangle=\varphi(x)\Psi(\varphi(x),t)$ and, therefore, acts as a multiplication operator, while the action of the canonical momentum operator is given by

$$\langle \varphi(x)|\hat{\pi}(x')|\Psi(t)\rangle = -i\hbar \frac{\delta}{\delta\varphi(x')}\langle \varphi(x)|\Psi(t)\rangle. \tag{2.9}$$

The Hamiltonian field operator $\hat{H}$ then becomes

$$\hat{H} = mc^2 \int \left[\frac{1}{2}\hat{\pi}^2(x) + \frac{1}{2}\left(\frac{\partial\hat{\varphi}}{\partial x}\right)^2 + (1-\cos\hat{\varphi})\right]dx, \tag{2.10}$$

where there is no ambiguity in the ordering and the functions of operators are defined by their power series.

The quantum mechanical problem for the field consists of solving the Schrödinger equation (2.7) for a given initial field configuration. Notice that the time in Eq. (2.7) has a scale set by $mc^2/\hbar$, which is of the same order of magnitude (10^{-23}) as the time scale l/c for the field fluctuations. This is to be expected since the Skyrme equation was assumed to be consistent with the quantum scale of the particle. Thus we can take the same dimensionless time variable for either scale.

III. APPROXIMATE SOLUTIONS TO THE FUNCTIONAL EQUATION

To solve the Schrödinger equation (2.7), we take a coherent state approximation [7–10] and study the evolution of its parameters. Following the approach in Refs. [7–10] we consider the functional

$$\Gamma = \int \langle \Psi | i\frac{\partial}{\partial t} - \hat{H} | \Psi \rangle dt, \tag{3.1}$$

which we extremize with the Gaussian trial functional

$$\Psi(\varphi(x),t) = \exp\left\{ i\int \pi(x,t)[\varphi(x)-\phi(x,t)]\; dx\right\} \times \exp\left\{-\int\int dx dy[\varphi(x)-\phi(x,t)] \times\left[\frac{1}{4}\Omega^{-1}(x,y,t) - i\Sigma(x,y,t)\right][\varphi(y)-\varphi(x)]\right\}, \tag{3.2}$$

where the kernels Ω^{-1} and Σ take into account the quantum fluctuations and $\phi(x,t)$ and $\pi(x,t)$ are the average field and average momentum of the Gaussian, respectively.

Substituting the trial function (3.2) into the functional (3.1) and integrating over the $\varphi(x)$ variable yields an averaged action [7–10]. This action is given in terms of $\phi(x,t)$, Ω and Σ. The potential is then also expanded around the average $\phi(x,t)$. It should be stressed that the consistency of this approximation depends on the smallness at all times of the variance Ω, in the sense that the average energy of the fluctuations around the mean $\phi(x,t)$ is small compared with the energy of the average motion. Moreover, by keeping only quadratic terms resulting from the averaging around the mean of the nonlinear term, which amounts to making the assumption that the energy of the fluctuations is small compared to the energy of the mean, we arrive at an effective action of the form

$$L = \int_0^T \left\{ \int_{-L}^{L}\left[\pi\frac{\partial\varphi}{\partial t} - \left(\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + (1-\cos\varphi)\right) + \Sigma\dot{\Omega}(x,x,t) - 2\Sigma\Omega\Sigma(x,x,t) - \cos(\varphi)\Omega(x,x,t) + \frac{1}{2}\frac{\partial^2\Omega}{\partial x^2}(x,y,t)|_{x=y} - \frac{1}{8}\Omega^{-1}(x,x,t)\right]\right\} dx dt + O(\Omega^2), \tag{3.3}$$

where the notation $\Omega\Sigma$ is taken to mean the kernel of the operator defined by the convolution of Σ with Ω, that is

$$\Omega\Sigma(x,y) = \int_{-L}^{L}\Omega(x,z)\Sigma(z,y)dz \tag{3.4}$$

and $\Sigma\Omega\Sigma$ is the kernel of the operator defined by the convolution of Σ, Ω and Σ

$$\Sigma\Omega\Sigma(x,y) = \int_{-L}^{L}\int_{-L}^{L}\Sigma(x,z)\Omega(z,u)\Sigma(u,y)dudz. \tag{3.5}$$

In the following we will take the limit $L\to\infty$ at different stages in the calculation of the effective action. The effective action can be computed once we choose an appropriate parametrization for the variance Ω and the phase Σ. It must be noted that thanks to the simple form of the potential. the Gaussian integral can be evaluated exactly. However, since we are interested in small fluctuations we stop at the quadratic level. We will come back to this point when comparing our results with those in [10].

Now, since the field is homogeneous, we can take

$$\Omega(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[ik(x-y)]\hat{\Omega}(k,t)dk \tag{3.6}$$

$$\Sigma(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[ik(x-y)]\hat{\Sigma}(k,t)dk, \tag{3.7}$$

where

$$\hat{\Omega}(k,t)=\frac{\Omega_0}{k^2+a^2(t)}, \tag{3.8}$$

$$\hat{\Sigma}(k,t)=\frac{b(t)}{k^2+a^2(t)}. \tag{3.9}$$

The parameters $a(t)$ and $b(t)$, which control the spreading of the fluctuations, are to be determined after the effective action is varied. The choice of the trial function is then guided by the simplicity of the resulting expressions. Notice, however, that since the results obtained depend only on the spreading, the same qualitative behavior will be obtained for other forms of the trial function.

Since the approximate solution (3.2) involves the kernel Ω^{-1} the proposed expression is convergent provided that the momentum of the fluctuations involved in the integration is low. This assumption is in agreement with the fact that the basic Skyrme model is consistent at low momentum and with the assumed homogeneity of the fluctuation. This is taken into account by taking for $\Omega^{-1}(x,y,t)$ the cut-off kernel

$$\Omega^{-1}(x,y,t)=\frac{1}{2\pi\Omega_0}\int_{-K}^{K}e^{ik(x-y)}(k^2+a^2)\ \ dk. \tag{3.10}$$

Then the non-constant contribution of $\Omega^{-1}(x,x,t)$ is

$$-\frac{1}{8}\Omega^{-1}(x,x,t)=-\frac{K}{8\pi\Omega_0}a^2. \tag{3.11}$$

In a similar manner we obtain

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\Omega(x,y,t)|_{y=x}=-\frac{1}{4\pi}\Omega_0\int_{-K}^{K}\frac{k^2}{k^2+a^2}dk$$
$$=\left[-\frac{\Omega_0 K}{2\pi}+\frac{1}{4\pi}a(t)\Omega_0\int_{-K/a}^{K/a}\frac{dk}{k^2+1}\right]. \tag{3.12}$$

The first term in this expression is infinite, but a constant. Thus it does not contribute to the equations of motion. We therefore take just the second term in the effective Lagrangian while noting at the same time that $K/a\gg 1$, since a is assumed to be small compared to K. Observe that the parameter Ω_0 measures the size of the fluctuations. In principle, other choices of the parameters may lead to different regularizations. However, as discussed below, the basic qualitative picture described in this work is not changed by these alternative regularizations, provided the momentum is low. For the case of higher momentum, procedures similar to the ones discussed in [8,10] can be used.

With the above assumptions, the effective Lagrangian takes the form

$$L=\int_0^T\int_{-L}^{L}\left\{\pi\frac{\partial\varphi}{\partial t}-\left(\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2+\left[1-\left(1-\frac{\Omega_0}{2\pi a}c_2\right)\cos\varphi\right]\right)\right.$$
$$+\frac{1}{2\pi}\left(-2\frac{\dot{a}b}{a^4}\Omega_0 c_1-2\frac{b^2}{a^5}\Omega_0 c_1-\frac{K}{4\Omega_0}a^2\right.$$
$$\left.\left.+\frac{1}{2}\Omega_0 c_2 a\right)\right\}\ \ dxdt, \tag{3.13}$$

where the constants c_1 and c_2 are given by

$$c_1=\int_{-\infty}^{\infty}\frac{dk}{(1+k^2)^3}=\frac{\pi}{2},\qquad c_2=\int_{-\infty}^{\infty}\frac{dk}{(1+k^2)}=\pi. \tag{3.14}$$

Note that c_2 approximates the integral in Eq. (3.12).

It is now convenient to change variables and define $q=1/a^3$ and $b=p$ in order to obtain the Lagrangian (3.13) in the form

$$L=\int_0^T\left\{\left(\frac{2c_1\Omega_0}{3}\dot{q}p-2c_1\Omega_0 p^2 q^{-5/3}-\frac{K}{4\Omega_0}q^{-2/3}\right.\right.$$
$$\left.+\frac{1}{2}\Omega_0\frac{c_2}{c_1}q^{-1/3}\right)\frac{L}{\pi}dt+\int_{-L}^{L}\left[\pi\frac{\partial\varphi}{\partial t}-\frac{1}{2}\pi^2-\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2\right.$$
$$\left.\left.-\left[1-\left(1-\frac{\Omega_0}{2}q^{1/3}\right)\cos\varphi\right]\right]dx\right\}dt. \tag{3.15}$$

The equations of motion are obtained by varying the Lagrangian (3.15) with respect to the parameters π, φ, p and q. These variational equations will consist of a partial differential equation for the field coupled to ordinary differential equations for the fluctuations. For the field the variational equation is

$$\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x^2}+\left(1-\frac{\Omega_0}{2}q^{1/3}\right)\sin\varphi=0. \tag{3.16}$$

The equations for p and q are derived from the Lagrangian

$$\mathcal{L}=\frac{2c_1 L\Omega_0}{3\pi}\int_0^T[\dot{q}p-3H(p,q)]dt, \tag{3.17}$$

where the Hamiltonian for the fluctuations is given by

$$H(p,q)=p^2q^{-5/3}+q^{1/3}\frac{1}{2L}\int_{-L}^{L}\cos\varphi\ \ dx-\frac{1}{8}q^{-1/3}$$
$$+\frac{K}{8c_1\Omega_0^2}q^{-2/3}. \tag{3.18}$$

The equations of motion for p and q are then given by Hamilton's equations as

$$\dot{q}=\frac{\partial H}{\partial p}=6pq^{-5/3} \tag{3.19}$$

$$\dot{p}=-\frac{\partial H}{\partial q}=5p^2q^{-8/3}+\frac{2}{3}\beta q^{-5/3}-\frac{1}{24}q^{-4/3} -\frac{q^{-2/3}}{2L}\int_{-L}^{L}\cos\varphi \quad dx, \tag{3.20}$$

where

$$\beta=\frac{3K}{8c_1\Omega_0^2}. \tag{3.21}$$

Since K is assumed to be small but $(K\gg a)$, we will take $\beta=2.5$ in the numerical calculations of the next section. It must be noted that for initial conditions which have $\varphi\to 2\pi$ as $x\to-\infty$ and $\varphi\to 0$ as $x\to\infty$, or vice versa,

$$\Lambda \coloneqq \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L}\cos\varphi\, dx=1. \tag{3.22}$$

It is therefore apparent that Eqs. (3.20) for the fluctuations decouple from Eq. (3.16) for the field.

It is interesting at this point to compare our equations (3.16), (3.19) and (3.20) with the corresponding Eqs. (4.5) of Ref. [10]. Observe that in that reference the equation for the field takes the form

$$\varphi_{tt}-\varphi_{xx}+\frac{\alpha_0}{\beta}e^{-(\gamma^2/2)G(x,x,t)}\sin\gamma\varphi=0. \tag{3.23}$$

Taking $\gamma=1$ and assuming $G(x,x,t)\ll 1$, which amounts to choosing the initial conditions in the form of small fluctuations, we have

$$e^{-(1/2)G(x,x,t)}\approx 1-\frac{1}{2}G(x,x,t). \tag{3.24}$$

Clearly, substituting this last expression into Eq. (3.23) recovers our Eq. (3.16), so the respective field equations agree for the above mentioned initial conditions and value of γ. Now, as for the second equation (4.5) in [10] note that this is an infinite system of partial differential equations for the operator valued function $\hat{\Psi}$, while in our formulation, because of the assumption of spatial homogeneity and the functional form chosen for the trial Green function, the system simplifies to a problem of ordinary differential equations. Another difference between our approach and that followed in [10] is that the equations proposed there in order to arrive at numerical solutions are integrodifferential equations, as opposed to the simple variational approximation we propose for obtaining solutions. It must be also remarked that our assumed homogeneity of the Green function is consistent with the low momentum limit we have chosen. In this low momentum limit, the fluctuations do not resolve the fine scale of the field and, to leading order, the configuration is a homogeneous background for the fluctuation. To conclude this section we also consider important to stress the fact that in previous published work the interest has been on static solutions allowing for arbitrary momentum of arbitrarily large fluctuations. This leads to a different renormalized version of the gap equation [7,8,10] and, for large coupling γ, to a loss of stability (phase transition). Our approximation does not capture this region, since we have assumed from the onset small fluctuations and small momentum. However it must be noted that our procedure could be extended to handle large momenta by choosing different trial functions for G and Σ, similar to the ones used in [8]. Also, due to the special form of the potential in the equations, the Gaussian integral may be evaluated to a better degree of approximation, thus allowing to include fluctuations of a larger amplitude. This program is currently under investigation and will be reported subsequently.

In the following section we undertake a detailed study of the dyamics described by Eqs. (3.16), (3.19) and (3.20).

IV. SOLUTIONS

The system of the sine-Gordon equation (3.16) and Eqs. (3.20) and (3.19) for q and p describe the coupling of the field to the fluctuations and the corresponding feedback. Note that the fluctuations have been assumed to be small. However, they are allowed to feed back onto the basic field configuration. We shall now use these equations to describe in a nonperturbative manner the nonlinear evolution of some special field configurations.

A. Quantum stability of the single soliton

We begin by studying the stability of the soliton solution (2.6) under a class of initial values for p and q. Note that a small value of q represents a small variance. Stability is then assured in the model by the fact that q remains small and that the field maintains its identity as a localized structure.

Numerical integrations of the sine-Gordon equation (3.16) and Eqs. (3.20) and (3.19) for p and q have been performed for a wide range of initial conditions and typical behaviors are shown in Figs. 1(a), 1(b), and 2. In Fig. 1(a) numerical solution for $\Phi=\varphi_x$ for the Skyrmion is shown and in Fig. 1(b) the behavior of a, the maximum of Φ, is shown. It can be seen that the fluctuations of the Skyrmion produce radiation, but that the field eventually stabilizes. This can be clearly seen from the behavior of the maxima shown in Fig. 2. The stabilization onto a modulated small oscillation of the Skyrmion amplitude can be clearly seen. These results exhibit the strong stability of the Skyrmion with respect to fluctuations. It is possible to understand this behavior by making use of the modulation theory given in [16] and [17] by means of the following argument. If the scales for p and q are slow, we may take as an approximate solution

$$\varphi=\varphi\left(\left(1-\frac{\Omega_0}{2}q^{1/3}\right)^{1/2}x\right), \tag{4.1}$$

which satisfies

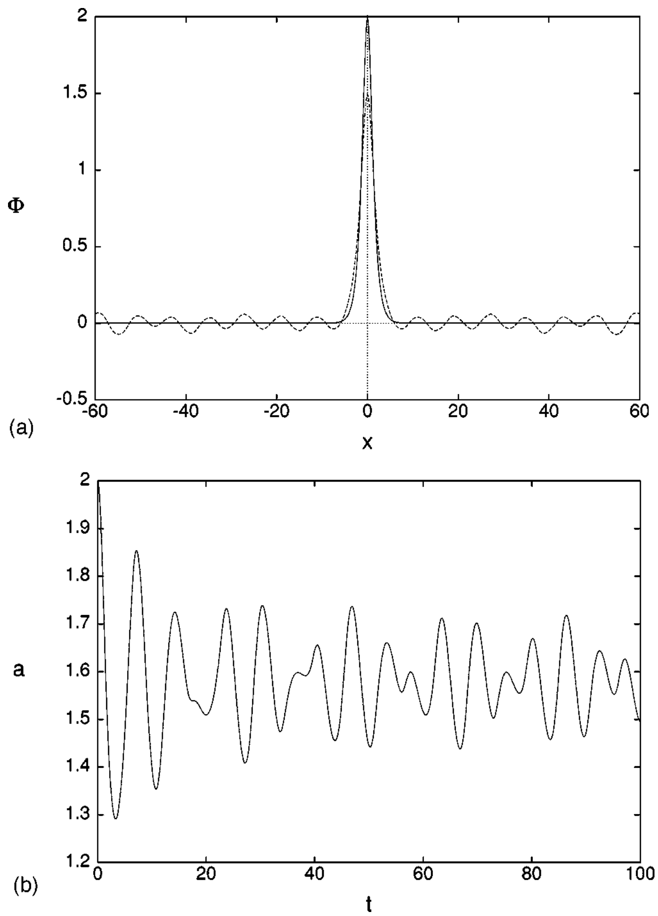

FIG. 1. Stability of a single soliton to fluctuations. Solution of sine-Gordon equation (3.16) and equations (3.20) for p and q with $\Omega_0=0.6$ and $\beta=2.5$. The initial conditions are $q=1.0$ and $p=0.0$ and $v=0$ in the soliton solution (2.6) at $t=0$. (a) soliton. — : initial condition; – – – : soliton at $t=100$. (b) Evolution of maximum a of $\Phi=\varphi_x$.

$$\cos\varphi=1-\frac{1}{2}\dot{\varphi}^2. \tag{4.2}$$

Hence as $L\rightarrow\infty$ we have from Eq. (3.22) that $\Lambda=1$. With this, the equations for p and q are derivable from the Hamiltonian

$$H(p,q)=3\left(p^2q^{-5/3}+q^{1/3}-\frac{1}{8}q^{-1/3}+\frac{K}{4\Omega_0^2}\ q^{-2/3}\right). \tag{4.3}$$

Thus the orbits in the (p,q) plane are just the level lines of

$$H(p,q)=E. \tag{4.4}$$

The orbits of the (p,q) system are then given by

$$\begin{aligned}p^2&=q^{5/3}\left[E-\left(\frac{K}{4\Omega_0^2}\ q^{-2/3}+q^{1/3}-\frac{1}{8}q^{-1/3}\right)\right]\\&=q^{5/3}[E-V(q)].\end{aligned} \tag{4.5}$$

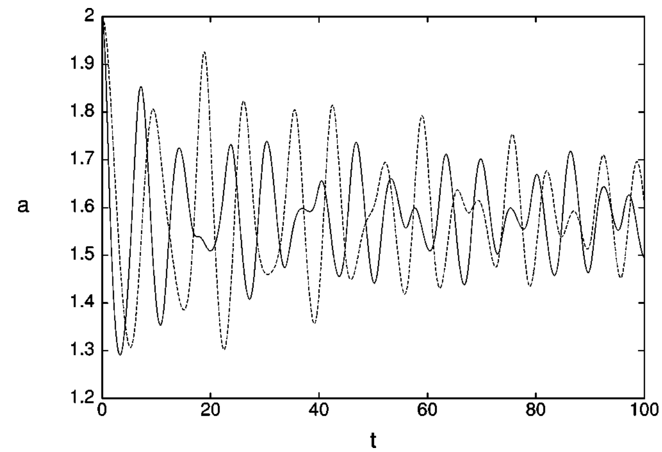

FIG. 2. Stability of a single soliton to fluctuations. Comparison between the full numerical solution of the sine-Gordon equation (3.16) and equations (3.20) for p and q and the approximate Eqs. (4.11). $\Omega_0=0.6$, $\beta=2.5$. The initial conditions are $q=1.0$, $p=0.0$ and $v=0$ at $t=0$. Amplitude a of $\Phi=\varphi_x$. Full numerical solution: — ; approximate solution: – – – .

The potential $V(q)$ has a minimum which gives an oscillatory solution for p and q, so that the width of the Skyrmion and thus the amplitude of $\Phi=\varphi_x$ oscillate in time. The numerical solution shown in Fig. 1(a) shows that the radiation, which is not taken into account in this approximation, stabilizes the oscillations onto a limit cycle.

This strongly nonlinear mechanism accounts for the stability of the Skyrmion. In fact it is the feedback of the field on the fluctuations which produces the term $q^{1/3}$ in $V(q)$ and it is this term which stabilizes the motion. The potential $V(q)$ has a maximum for small q. For energies E larger than this maximum, the fluctuations q increase and the field structure is destroyed. However the value of q for this to occur is too small to be consistent with the coherent state approximation. The model is therefore self-consistent and provides an explanation of how nonlinear interactions are responsible for the quantum stability of the field.

The approximate solution above does not take into account the radiation produced by the oscillating Skyrmion and so this approximate solution will not give the baryon settling onto a limit cycle solution. To take account of the radiation the ideas of Smyth and Worthy [18] can be used. In this work the effect of shed dispersive radiation on the evolution of a single pulse for the sine-Gordon equation was treated. To take account of the radiation we proceed as in [18], indicating only the main differences from this work.

The Lagrangian density for the sine-Gordon equation is

$$L=\frac{1}{2}\varphi_t^2-\frac{1}{2}\varphi_x^2-\left(1-\frac{\Omega_0}{2}q^{1/3}\right)(1-\cos\varphi). \tag{4.6}$$

To obtain an approximate solution of the sine-Gordon equation, the trial function

$$\varphi=-4\arctan e^{-x/w(t)}, \tag{4.7}$$

which is a solitonlike pulse with varying width $w(t)$, is substituted into the averaged Lagrangian

$$\overline{L}=\int_{-\infty}^{\infty}L\,dx=\frac{\pi^2}{3}\frac{w'^2}{w}-\frac{4}{w}-4\left(1-\frac{\Omega_0}{2}q^{1/3}\right)w, \quad (4.8)$$

as in [18]. In this approximation the Hamiltonian for p and q again does not change due to Eq. (3.22).

The effect of the radiation shed by the evolving soliton is determined by finding an appropriate solution of the linearized sine-Gordon equation [18]. The effect of this radiation is then to modify the Euler-Lagrange equation for $w(t)$. It is noted from the numerical solution of Fig. 1(a) that the radiation $\tilde{\varphi}$ is of small amplitude compared with the soliton. Therefore following [18] we consider the linearized sine-Gordon equation

$$\frac{\partial^2\tilde{\varphi}}{\partial t^2}-\frac{\partial^2\tilde{\varphi}}{\partial x^2}+\left(1-\frac{\Omega_0}{2}q^{1/3}(t)\right)\tilde{\varphi}=0 \quad (4.9)$$

for the radiation $\tilde{\varphi}$. This equation is solved together with appropriate source conditions at the pulse at $x=0$. Since $\Omega_0\ll 1$,

$$\frac{d}{dt}\left(1-\frac{\Omega_0}{2}q^{1/3}(t)\right)\ll 1. \quad (4.10)$$

It is then possible to obtain an expression for the radiation by making the adiabatic approximation that $1-(\Omega_0/2)q^{1/3}$ is constant to leading order. The effect of the radiation can then be found from the expression of [18] by a suitable re-scaling. In this manner we find that the equations governing the evolution of the soliton, including the effect of radiation, are

$$\frac{2\pi^2}{3w}\frac{d^2w}{dt^2}-\frac{\pi^2}{3w^2}\left(\frac{dw}{dt}\right)^2-\frac{4}{w^2}+4\left(1-\frac{\Omega_0}{2}q^{1/3}\right)=\frac{1}{\sqrt{\lambda}}\left[-\frac{2\pi^2}{3w\sqrt{\lambda}\ \ t}\frac{dw}{dt}+\frac{2\pi^2}{3\sqrt{wt}}\int_0^t J_1(\sqrt{\lambda}(t-\tau))\frac{w'(\tau)}{\sqrt{\tau w(\tau)}}\,d\tau\right]$$

$$\frac{dq}{dt}=6pq^{-5/3} \quad (4.11)$$

$$\frac{dp}{dt}=5p^2q^{-8/3}+\frac{2}{3}\beta q^{-5/3}-q^{-2/3}+\frac{1}{9}q^{-4/3},$$

where

$$\lambda=1-\frac{\Omega_0}{2}q^{1/3}. \quad (4.12)$$

These equations were integrated numerically. Comparisons between solutions of these equations and the full numerical solution of the sine-Gordon equation for the amplitude a of $\Phi=\varphi_x$ and $q(t)$ for the fluctuations are shown in Fig. 2. It can be seen also that the amplitude agreement shown in Fig. 2 is good considering the assumptions that were made to incorporate the radiation loss in the approximate equations. It can be seen that the approximate equations provide a suitable approximate solution for the full field behavior using a finite dimensional approximation which includes radiation. Note that, since $q(t)$ is periodic, the sine-Gordon equation (3.16) is subject to a parametric excitation. However the nonlinearity and radiation loss provide the necessary damping to enable a limit cycle to be achieved.

B. The collision of a wave with a static soliton

As a final example we consider the scattering of a wave packet representing a pion with momentum k with a static soliton, representing a baryon originally at rest. The problem is solved by numerically integrating the sine-Gordon equation (3.16) using the initial condition

$$\varphi(x)=-4\arctan e^{-x}+f(x) \quad (4.13)$$

$$\frac{\partial\varphi}{\partial t}=g(x), \quad (4.14)$$

where the functions f and g are given by

$$f(x)=a\sin k(x+x_0), \quad |x+x_0|\leqslant\delta \quad (4.15)$$

$$g(x)=-a\sqrt{k^2+1}\cos k(x+x_0), \quad |x+x_0|\leqslant\delta. \quad (4.16)$$

This initial condition represents an incoming meson with momentum k impinging on a nucleon located at $x=0$. A numerical solution for the scattering of the pion wave packet can be seen in Fig. 3. The initial condition (at $t=0$) is shown by the solid line in Fig. 3(a). In this figure a reflected wave packet, a reorganized field configuration and a new packet shed by the baryon as a result of the interaction can be seen. In Figs. 3(a), 3(b), and 3(c), the complicated evolution of the baryon amplitude is displayed. This amplitude behavior is due to the interaction of the baryon with the packet. The scattering then involves a reorganization of the field, which is not taken into account when the particles are taken to be point particles. The description of the interaction of the baryon with radiation using a multi-phase solution of the sine-Gordon equation is under investigation at present.

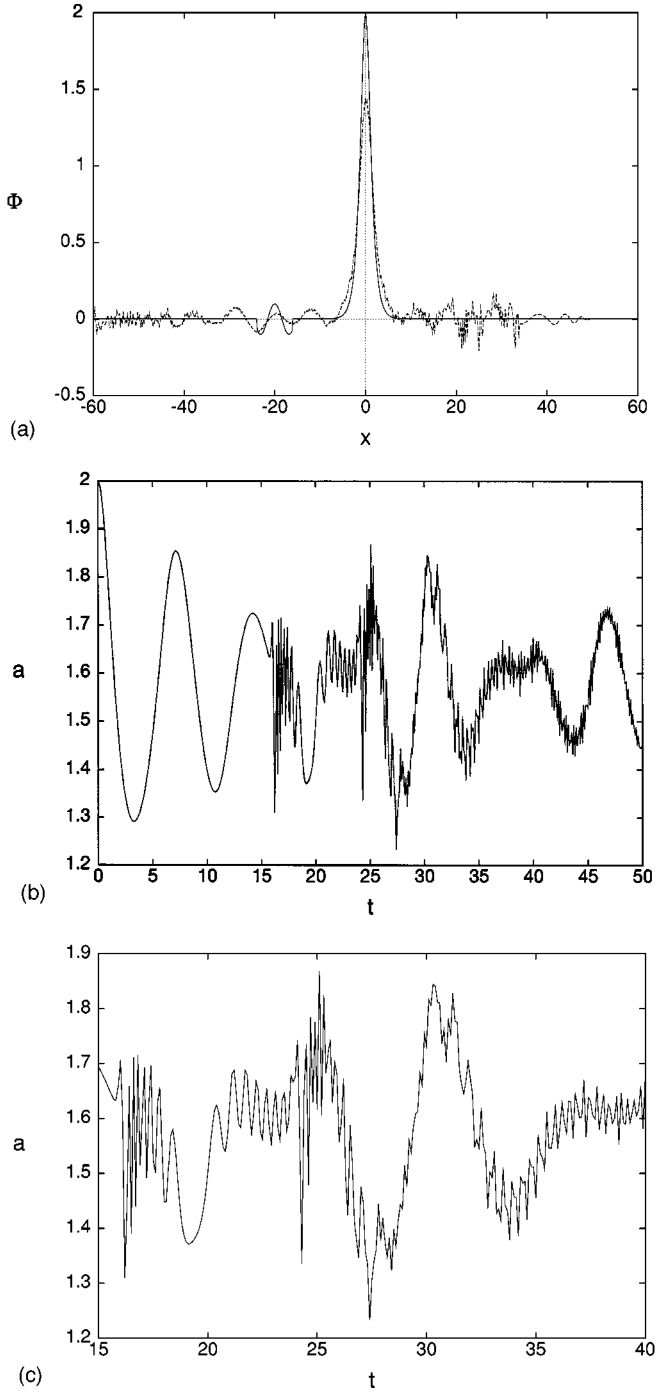

FIG. 3. Scattering of a wave packet (pion) with a baryon. Solution of sine-Gordon equation (3.16) and equations (3.20) for p and q with $\Omega_0 = 0.6$ and $\beta = 2.5$. The initial conditions are given by Eqs. (4.13),(4.14) and $a = 0.1$, $k = 1.0$, $\delta = 4.0$ and $x_0 = 20$ in Eqs. (4.15) and (4.16). Also $q = 1.0$ and $p = 0.0$ at $t = 0$. (a) Solution at $t = 50$. (b) Evolution of maximum a of $\Phi = \varphi_x$. (c) Detail of evolution of maximum a of $\Phi = \varphi_x$ from $t = 15$ to $t = 40$.

C. Collision of two solitons in the presence of a fluctuation

Since the classical field equation is completely integrable, solitons interact elastically and do not change configuration. The effect of quantum fluctuations on the collisions of solitons and this clean interaction will now be studied.

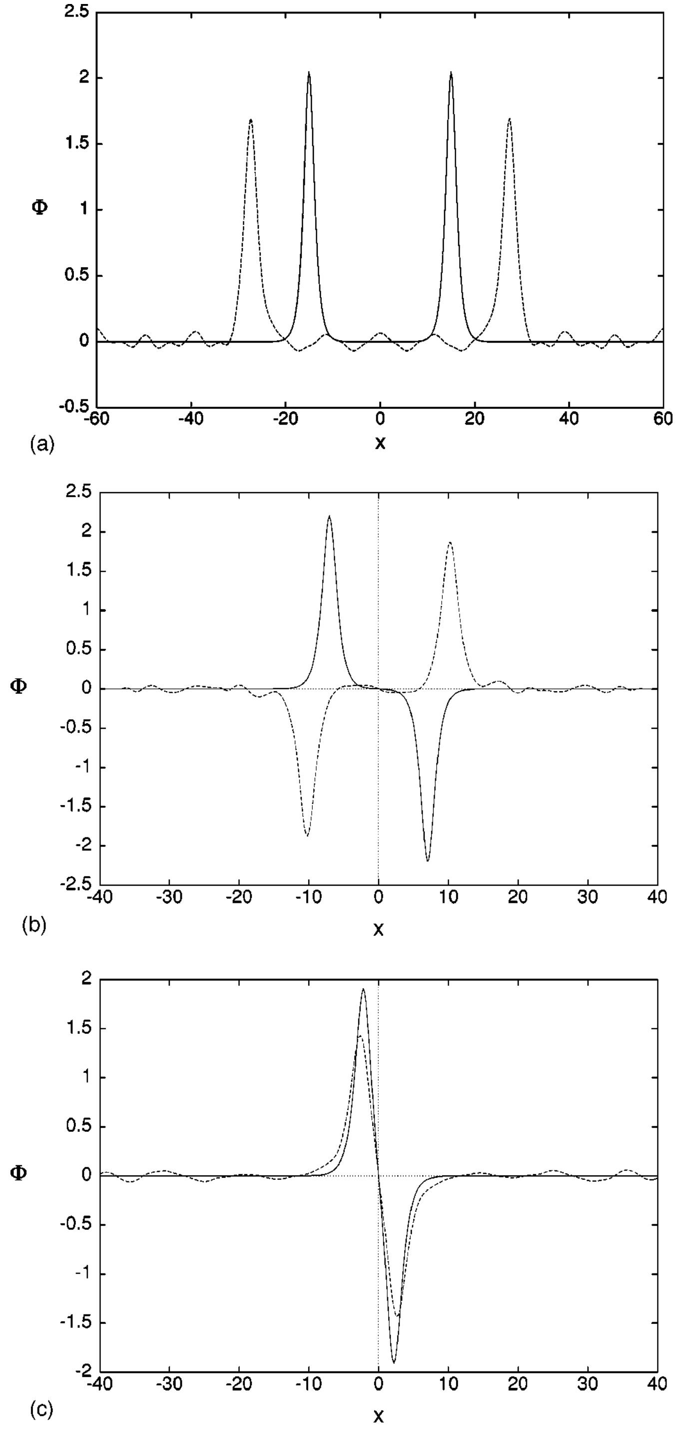

FIG. 4. Collisions of solitons. Initial conditions have $q = 1.0$ and $p = 0.0$. $\Omega_0 = 0.6$. (a) Two solitons. Initial condition (4.17) with $x_0 = 15$ and $v = 0.2$. Initial condition $(t = 0)$: —; solution at $t = 150$: – – – – . (b) Soliton and an anti-soliton. Initial condition (4.18) with $\alpha = 1.2$. Initial condition $(t = -15)$: — ; solution at $t = 15$: – – – – . (c) Bound state of a soliton and an anti-soliton. Initial condition (4.19) with $\alpha = 0.98$. Initial condition $(t = -5)$: — ; solution at $t = 45$: – – – –.

Figure 4(a) shows the collision of two solitons with equal and opposite velocity. The initial condition used was

$$\varphi = 2\pi - 4\arctan e^{-(x+x_0-vt)/\sqrt{1-v^2}} - 4\arctan e^{-(x-x_0+vt)/\sqrt{1-v^2}} \qquad (4.17)$$

as $t \to -\infty$. Since there is no classical solution with twice the baryon number and zero velocity, the effect of the quantum fluctuations is just to slightly modify the classical interaction. The solitons again settle down to a limit cycle for which the parametric resonance is balanced by the radiation damping.

Figure 4(b) shows the collision of a soliton and an anti-soliton. The initial condition is

$$\varphi = -4\arctan\left[\frac{\alpha}{\sqrt{\alpha^2-1}}\,\frac{\sinh\sqrt{\alpha^2-1}\;\; t}{\cosh \alpha x}\right]. \qquad (4.18)$$

Again this interaction does not produce disintegration, just a modification of the classical interaction.

Finally the susceptibility to disintegration of the breather-type configuration

$$\varphi = -4\arctan\left[\frac{\alpha}{\cosh \alpha x}\,\frac{\sin\sqrt{1-\alpha^2}\;\; t}{\sqrt{1-\alpha^2}}\right] \qquad (4.19)$$

with frequency $\sqrt{1-\alpha^2}$ is studied. From the numerical solution shown in Fig. 4(c) it can be seen that the breather is stable with respect to quantum fluctuations.

The solutions displayed in Fig. 4 show that the reduction of the Skyrme model to the sine-Gordon equation is too severe for treating collisions. In order to obtain non-trivial collision and fusion processes, such as those possible for the nonlinear Schrödinger equation, reductions of the Skyrme model which retain more internal degrees of freedom must be derived.

V. CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

We have formulated the quantum field problem for the sine-Gordon equation which is related to the (dimensionally reduced) Skyrme model. Using the coherent state approximation for the solution of the functional Schrödinger equation, we obtain a solution of the partial differential equation for the (quantum corrected, semi-classical) field, which is coupled to ordinary differential equations for the fluctuations. Other quantizations for nonlinear fields keep only finitely many degrees of freedom (minisuperspace approximation), which are then quantized in a canonical way.

The first problem considered in the present work was the stability with respect to quantum fluctuations of a soliton. Both numerical and asymptotic solutions were considered. It was found that the nonlinear saturation of the field equation together with the loss of radiation balanced the parametric excitation of the fluctuations. The fluctuations in turn were controlled by the shape of the field. The good agreement found between numerical and asymptotic solutions suggests that finite dimensional approximations to the dynamics of the Skyrmion model, such as those used in [6], are also good approximations to the full dynamics of more complicated problems, such as those treated there.

The scattering of a wave by a static soliton was also studied. The numerical results obtained show well defined waves and a Skyrmion after collision, which suggests the possibility of using multi-phase solutions, such as those of [19], to understand this scattering process.

Finally several collision processes were studied. It was found that the reduced Skyrme model cannot account for the collision and fusion of baryons. Therefore the study of the fusion of Skyrmions into a toroidal configuration requires a uniform solution which interpolates between the torus and the individual Skyrmions. The possibility of using the solutions given in [20,21] is currently under study. It must be noted that more sophisticated numerical formulations such as the ones proposed in [10] must produce, in the limit of low momentum, solutions comparable to our results.

To conclude, we note that the techniques described in this work can be applied to the study of low dimensional black holes. Indeed, an old observation that the sine-Gordon theory and 2-dimensional spaces of constant curvature are very closely related has recently found an interesting application to gravity in $1+1$ dimensions. More precisely, Gegenberg and Kunstatter [22] have noticed that when a two dimensional Lorenzian metric is parametrized as

$$ds^2 = -\sin^2(u/2)dt^2 + \cos^2(u/2)dx^2, \qquad (5.1)$$

then the condition of constant curvature is equivalent to the condition that u satisfies the *Euclidean* sinh-Gordon equation. On the other hand, the so-called Jackiw-Teitelboim theory in two dimensions

$$I = \int \phi(R-\Lambda)\sqrt{-g}\;\; dt dx \qquad (5.2)$$

has as solutions space-times of constant curvature $R=\Lambda$. Furthermore, the one-soliton solution of the sine-Gordon equation has been found to represent (a patch of) a black hole solution of the (Jackiw-Teitelboim) theory [22]. That a constant curvature space-time can be interpreted as a black hole is not unique to two dimensions. The $1+1$ Jackiw-Teitelboim black hole can indeed be interpreted as a dimensionally reduced Bañados-Teitelboim-Zanelli (BTZ) (non-rotating) black hole and many of its properties (including thermodynamics) have been studied [23].

To perform an analysis similar to the one presented in the present work for the Euclidean sine-Gordon equation is cumbersome, since the equation is now elliptic and does not accept a well-posed initial value formulation. However it is possible to work in the framework of a well-posed problem if one chooses a different parametrization for the two dimensional space-time as follows:

$$ds^2 = -\sinh^2(u/2)dt^2 + \cosh^2(u/2)dx^2. \qquad (5.3)$$

In this case, the constant curvature condition reduces to the *Lorenzian* sine-Gordon equation. It is then possible to analyze the quantum stability of a black hole solution using the functional methods presented in this article. This work will be reported elsewhere.

As a final remark we point out that the quantum equations for a classical field obtained using the functional Schrödinger equation and the coherent state approximation will always have the same structure. Namely the classical equations for the field with renormalized (fluctuating) parameters and equations for the (parameters of the) fluctuations which are non-local in the fields will always be obtained.

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