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Complex Ginzburg–Landau equations as perturbations of nonlinear Schrödinger equations A Melnikov approach

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Abstract

We study the persistence of quasiperiodic and homoclinic solutions of generalized nonlinear Schrödinger equations under Ginzburg–Landau perturbations. In this paper, the first of a series, Melnikov criteria for the persistence of quasiperiodic and homoclinic solutions are derived directly from the governing partial differential equations via an averaging technique. For families of tori of quasiperiodic solutions, such as rotating waves and traveling waves, that arise within critical sets of linear combinations of conserved functionals, we find that usually only isolated tori will satisfy these selection criteria. Moreover, in some simple cases these criteria are sufficient to conclude that a torus persists. We also demonstrate the nonpersistence of solutions that are homoclinic to rotating waves under a broad class of Ginzburg–Landau perturbations which satisfy a convexity condition.

Keywords: Quasiperiodic; Homoclinic; Melnikov criteria

1. Introduction

A central problem in the study of a dynamical system is the identification of its long-time dynamics. Many infinite-dimensional dissipative systems possess a global attractor that captures this dynamics. When such a system is a damped-driven perturbation of a Hamiltonian system, it is natural to ask whether objects in its attractor are

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perturbations of objects in its unperturbed Hamiltonian phase space. In other words, one can ask what objects in its unperturbed Hamiltonian phase space persist under the perturbation. In particular, one can ask this question about families of special solutions of the Hamiltonian system that either are realized through critical sets of linear combinations of conserved quantities (such as traveling waves) or are homoclinic to such critical sets. Most of the solutions in such a family will not persist when, as typically happens, the perturbation destroys the underlying conservation properties. This paper initiates a series [2–4] in which we seek to identify those solutions that do persist through a Melnikov approach that utilizes those conserved quantities associated with a given family's construction.

As part of our general program, we consider the persistence of solutions of the generalized nonlinear Schrödinger (GNLS) equation

$$\partial_t A = \mathrm{i}\partial_{xx} A - \mathrm{i}h'(|A|^2)A, \tag{1.1}$$

when the GNLS equation is perturbed to a generalized complex Ginzburg–Landau (GCGL) equation

$$\partial_t A = (\mathrm{i}+\epsilon)\partial_{xx} A - \mathrm{i}h'(|A|^2)A - \epsilon g'(|A|^2)A. \tag{1.2}$$

Here $A(x, t)$ is a complex-valued field, $\epsilon > 0$, while $h = h(\xi)$ and $g = g(\xi)$ are real analytic functions over $[0, \infty)$. For example, common realizations are power law nonlinearities of the form

$$h(\xi) = \pm\frac{2}{s}\xi^s, \qquad g(\xi) = -r\xi + \frac{2q}{p}\xi^p, \tag{1.3}$$

where s and p are integers greater than one, and r and q are positive constants. The $\pm$ selects the so-called defocusing $(+)$ or focusing $(-)$ case. While we could have considered a more general form than (1.2) for the GCGL equation, we chose the form (1.2) because it is general enough to show the utility of the methods contained herein while special enough to keep the number of technicalities manageable. Finally, we consider $A(x, t)$ to be periodic in x with period one, whereby we consider x in $\boldsymbol{T} \equiv \boldsymbol{R}/\boldsymbol{Z}$.

Considered as an initial-value problem over $H^1(\boldsymbol{T})$, the GCGL equation (1.2) is well-posed locally in time. These local solutions are real analytic, and hence classical, for positive times so long as they exist [5,7]. Moreover, under certain conditions it can be shown that the GCGL equation is globally well-posed in $H^1(\boldsymbol{T})$ for every $\epsilon > 0$. This is known to be the case [7] for the power law nonlinearities (1.3) when $s = p$ in the defocusing case, or when $s = p = 2$ or 3 in the focusing case. More generally, by similar arguments one can show that (1.2) is globally well-posed for every $\epsilon > 0$ when h and g are convex functions (in particular, without the restriction $s = p$ when in the defocusing case of (1.3)). There are many other cases that could be listed for which global well-posedness can be established for every $\epsilon > 0$, but that would not serve our purpose. Rather, we assume that we are in such a case, so that the GCGL equation defines a dynamical system over $H^1(\boldsymbol{T})$. Moreover, we assume that this system has a compact global attractor, that is, a connected, invariant set that attracts uniformly over bounded sets of initial data. For example, for the power law nonlinearities (1.3) either when $s = p$ in the defocusing case, or when $s = p = 2$ or 3 in the focusing case, the GCGL dynamics has a compact global attractor which attracts uniformly over all initial data [7]. In this case of power law nonlinearities it has been shown, using estimates on the energy functional of the NLS equation and the Gagliardo–Niremberg inequalities, the weak convergence of solutions of the CGL equation to solutions of the NLS equation [18,16].

An important special case, about which much more is known, arises when $s = p = 2$ for the power law nonlinearities (1.3). This yields the case wherein the celebrated cubic NLS (cNLS) equation,

$$\partial_t A = \mathrm{i}\partial_{xx} A \mp \mathrm{i}2|A|^2 A, \tag{1.4}$$

is perturbed to the ubiquitous cubic Ginzburg–Landau (cCGL) equation,

$$\partial_t A = \epsilon r A + (\epsilon + \mathrm{i})\partial_{xx} A - 2(\epsilon q \pm \mathrm{i})|A|^2 A. \tag{1.5}$$

In both the defocusing and focusing cases, the cNLS equation is a completely integrable Hamiltonian system via the inverse spectral method [8–11]. The phase space $H^1(\boldsymbol{T})$ of the defocusing cNLS equation is foliated by neutrally stable invariant tori of solutions that are almost periodic in time. The phase space of the focusing cNLS decomposes into invariant tori with connecting homoclinic and heteroclinic orbits. Moreover, global attractors of the defocusing and focusing cCGL dynamics are contained in inertial manifolds (finite-dimensional manifolds that are exponentially attracting and positively invariant under the flows (cf. [12–14])) and consequently display low-dimensional long-time behavior [15,17,19–21].

The problem of establishing connections between objects in the global attractor of the GCGL equation and special solutions of the GNLS equation has a long history. Some numerical studies suggested that the cubic CGL (cCGL) solutions appeared to become simple (often just periodic) after what was sometimes a very long transient. In [22] the resulting spatial profiles for some value of ϵ were then used as initial data for the cCGL equation with a smaller value of ϵ. These solutions were quickly attracted to a solution with a very similar spatial profile. This process was repeated a few times until ϵ was quite small and the resulting spatial profiles were analyzed by a numerical inverse spectral transform. These profiles were thereby identified as deformations of certain cNLS solutions. The study therefore suggested that at least some cCGL solutions have ω-limit sets that deform to cNLS solutions as ϵ tends to zero.

The question we pose here is as follows. For a given choice of $h(\xi)$ and $g(\xi)$, and assuming that some solutions of the GNLS equation (4.26) for the given choice of $h(\xi)$ are known, which of these solutions *persists* when the perturbation is turned on ($0 < \epsilon \ll 1$)? The notion of persistence for general tori of quasiperiodic solutions will be precisely defined in Section 3.

The next papers in this series [2–4] expand related studies developed by the authors [1,23,6,24]. In particular, in [3] we will present detailed results for the cCGL case as a concrete example of our methods. Other references to works that study the long-time dynamics of the Ginzburg–Landau equation can be found in these papers.

Other researchers have studied persistence questions related to the cNLS equation, most notably the extensive program undertaken by McLaughlin et al., who have studied the structure of the focusing cNLS equation in detail and the persistence of homoclinic solutions [25,26]. The type of perturbation considered in those papers differs from that considered here in that it does not preserve the phase symmetry of the GNLS equation (invariance under $A \mapsto e^{i\tau} A$) and therefore this type of perturbation does not hold condition Section 2.13 given ahead. In our case, it is important to hold this condition to preserve some of the invariate tori of the unperturbed problem.

The rest of this paper is organized as follows. We lay out the Hamiltonian structure of the GNLS equation that underlies our analysis in Section 2. We give a precise definition for the notion of persistence for a torus of quasiperiodic (in time) solutions in Section 3, and derive necessary conditions for the persistence of such a torus of GNLS solutions under the GCGL perturbation. In Section 4 we show that these conditions are sometimes sufficient for a torus to persist. In particular, this question is examined for the rotating wave and traveling wave GNLS solutions. In Section 5 we turn our attention to homoclinic solutions. We define the notion of persistence for solutions homoclinic to a torus of traveling wave solutions and derive necessary conditions for the persistence of such GNLS solutions under the GCGL perturbation. In Section 6 we then prove that all GNLS solutions homoclinic to rotating waves are destroyed by GCGL perturbations when the nonlinearities of the equations satisfy a convexity condition. We also present numerical evidence that the GCGL perturbation can introduce new homoclinic orbits in the cubic case which are structurally different from the homoclinics of the unperturbed problem.

2. Hamiltonian structure

The GNLS equation (1.1) can be put in the abstract Hamiltonian form

$$\partial_t A = -i \frac{\delta \mathcal{H}}{\delta A^*}, \tag{2.1}$$

where the Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H}(A) \equiv \int_0^1 (|\partial_x A|^2 + h(|A|^2))\,\mathrm{d}x. \tag{2.2}$$

The evolution of any real-valued functional $\mathcal{F}$ under the flow governed by (2.1) formally obeys

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = \{\mathcal{F}, \mathcal{H}\}, \tag{2.3}$$

where the Poisson bracket of any two functionals $\mathcal{F}_1$ and $\mathcal{F}_2$ is defined by

$$\{\mathcal{F}_1, \mathcal{F}_2\} \equiv -\mathrm{i}\int_0^1 \left(\frac{\delta\mathcal{F}_1}{\delta A}\frac{\delta\mathcal{F}_2}{\delta A^*} - \frac{\delta\mathcal{F}_1}{\delta A^*}\frac{\delta\mathcal{F}_2}{\delta A}\right)\mathrm{d}x. \tag{2.4}$$

Two functionals whose Poisson bracket vanishes are said to Poisson commute.

By (2.3), the class of functionals conserved by the GNLS flow (2.1) are precisely those that Poisson commute with $\mathcal{H}$. Besides $\mathcal{H}$, when $\mathcal{H}$ is given by (2.2) the GNLS flow also conserves at least two quantities, the mass $\mathcal{N}$ and the momentum $\mathcal{J}$ given by

$$\mathcal{N}(A) \equiv \int_0^1 |A|^2\,\mathrm{d}x, \qquad \mathcal{J}(A) \equiv \frac{1}{2\mathrm{i}}\int_0^1 (A^*\partial_x A - A\partial_x A^*)\,\mathrm{d}x. \tag{2.5}$$

Indeed, these quantities satisfy

$$\{\mathcal{N}, \mathcal{J}\} = \{\mathcal{N}, \mathcal{H}\} = \{\mathcal{J}, \mathcal{H}\} = 0, \tag{2.6}$$

i.e. they mutually Poisson commute. This means the Hamiltonian flows they generate mutually commute as well. If we let t_0 and t_1 denote the times associated with the Hamiltonian flows generated by $\mathcal{N}$ and $\mathcal{J}$, these flows are

$$\partial_{t_0} A = -\mathrm{i}\frac{\delta\mathcal{N}}{\delta A^*} = -\mathrm{i}A, \qquad \partial_{t_1} A = -\mathrm{i}\frac{\delta\mathcal{J}}{\delta A^*} = -\partial_x A. \tag{2.7}$$

These flows are just phase rotation and spatial translation on A, respectively. By (2.6), if $A(x, t)$ solves the GNLS then so does $\mathrm{e}^{-\mathrm{i}t_0}A(x - t_1, t)$ for any (t_0, t_1), reflecting the symmetry of the GNLS dynamics under these flows.

We will consider real-valued functionals $\mathcal{F}$ that are smooth functions of A over $C^\infty(\boldsymbol{T})$. For example, $\mathcal{F}$ could have the form

$$\mathcal{F}(A) = \int_0^1 f(A, A^*, \partial_x A, \partial_x A^*, \ldots, \partial_x^k A, \partial_x^k A^*)\,\mathrm{d}x, \tag{2.8}$$

where f is an analytic function of its arguments. This ensures that the above formal calculations make sense provided A is C^∞. In fact, in the calculations that follow A will be real-analytic. For example, in the cubic NLS (cNLS) case there exist an infinite set of conserved functional with these properties.

We will consider the evolution under the GCGL flow of real-valued functionals that are conserved by the GNLS flow. More generally, the GCGL equation (1.2) can be put in the abstract form

$$\partial_t A = -\mathrm{i}\frac{\delta\mathcal{H}}{\delta A^*} - \epsilon\frac{\delta\mathcal{G}}{\delta A^*}, \tag{2.9}$$

where $\mathcal{H}$ is given by (2.2) and the Ginzburg–Landau functional $\mathcal{G}$ is given by

$$\mathcal{G}(A) \equiv \int_0^1 (|\partial_x A|^2 + g(|A|^2))\,\mathrm{d}x. \tag{2.10}$$

The evolution of any real-valued functional $\mathcal{F}$ that Poisson commutes with $\mathcal{H}$ when evaluated along a solution of (2.9) formally obeys

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = \int_0^1 \left(\frac{\delta \mathcal{F}}{\delta A^*} \partial_t A^* + \frac{\delta \mathcal{F}}{\delta A} \partial_t A \right) \mathrm{d}x = \int_0^1 \left(\frac{\delta \mathcal{F}}{\delta A^*} \left(\mathrm{i} \frac{\delta \mathcal{H}}{\delta A} - \epsilon \frac{\delta \mathcal{G}}{\delta A} \right) + \frac{\delta \mathcal{F}}{\delta A} \left(-\mathrm{i} \frac{\delta \mathcal{H}}{\delta A^*} - \epsilon \frac{\delta \mathcal{G}}{\delta A^*} \right) \right) \mathrm{d}x$$
$$= -\epsilon \int_0^1 \left(\frac{\delta \mathcal{F}}{\delta A^*} \frac{\delta \mathcal{G}}{\delta A} + \frac{\delta \mathcal{F}}{\delta A} \frac{\delta \mathcal{G}}{\delta A^*} \right) \mathrm{d}x.$$

The last expression does not involve $\mathcal{H}$ because $\{\mathcal{F}, \mathcal{H}\} = 0$. This becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}(A(t)) = -\epsilon \mathcal{S}^{\mathcal{F}}(A(t)), \tag{2.11}$$

where for any A in $C^\infty(\boldsymbol{T})$, the functional $\mathcal{S}^{\mathcal{F}}$ is defined by

$$\mathcal{S}^{\mathcal{F}}(A) \equiv \int_0^1 \left(\frac{\delta \mathcal{F}}{\delta A^*} \frac{\delta \mathcal{G}}{\delta A} + \frac{\delta \mathcal{F}}{\delta A} \frac{\delta \mathcal{G}}{\delta A^*} \right) (A) \, \mathrm{d}x. \tag{2.12}$$

We will apply the above calculations to solutions in the GCGL global attractor, which we have assumed to exist (as in the case of power nonlinearities explained below). The calculations will therefore be justified because every such solution is real-analytic.

When $\mathcal{G}$ is given by (2.10), the GCGL flow, like the GNLS flow, commutes with those of phase rotation and spatial translation (2.7) because

$$\{\mathcal{G}, \mathcal{N}\} = \{\mathcal{G}, \mathcal{J}\} = 0, \tag{2.13}$$

where $\mathcal{N}$ and $\mathcal{J}$ are given by (2.5). We will therefore consider only those functionals $\mathcal{F}$ that are conserved by the flows of phase rotation and spatial translation as well as that of GNLS, namely those that satisfy

$$\{\mathcal{F}, \mathcal{N}\} = \{\mathcal{F}, \mathcal{J}\} = \{\mathcal{F}, \mathcal{H}\} = 0. \tag{2.14}$$

From (2.13), (2.14) and (2.12) it follows that $\mathcal{S}^{\mathcal{F}}$ is also conserved by the flows of phase rotation and spatial translation, namely one has

$$\{\mathcal{S}^{\mathcal{F}}, \mathcal{N}\} = \{\mathcal{S}^{\mathcal{F}}, \mathcal{J}\} = 0. \tag{2.15}$$

Indeed, that $\mathcal{S}^{\mathcal{F}}$ is conserved by phase rotation follows from the fact that by (2.13) and (2.14) the same is true for $\mathcal{F}$ and $\mathcal{G}$. This means that $\mathcal{F}(\mathrm{e}^{-\mathrm{i}t_0} A) = \mathcal{F}(A)$ and $\mathcal{G}(\mathrm{e}^{-\mathrm{i}t_0} A) = \mathcal{G}(A)$ for every real t_0, which implies that

$$\frac{\delta \mathcal{F}}{\delta A^*}(\mathrm{e}^{-\mathrm{i}t_0} A) = \mathrm{e}^{-\mathrm{i}t_0} \frac{\delta \mathcal{F}}{\delta A^*}(A), \qquad \frac{\delta \mathcal{G}}{\delta A^*}(\mathrm{e}^{-\mathrm{i}t_0} A) = \mathrm{e}^{-\mathrm{i}t_0} \frac{\delta \mathcal{G}}{\delta A^*}(A). \tag{2.16}$$

Definition (2.12) shows that $\mathcal{S}^{\mathcal{F}}$ is just a real inner product of $\delta\mathcal{F}/\delta A^*$ with $\delta\mathcal{G}/\delta A^*$. Because $\mathrm{e}^{-\mathrm{i}t_0}$ is a unitary operator with respect to this inner product, by (2.16) it follows that $\mathcal{S}^{\mathcal{F}}(\mathrm{e}^{-\mathrm{i}t_0} A)$ is independent of t_0. Thus, $\mathcal{S}^{\mathcal{F}}$ is conserved by phase rotation. A similar argument with $\mathrm{e}^{-\mathrm{i}t_0}$ replaced by the translation operator $\mathrm{e}^{-t_1 \partial_x}$ establishes that $\mathcal{S}^{\mathcal{F}}$ is conserved by spatial translation.

3. Necessary criteria for the persistence of tori

In this section, we define the notion of persistence as applied to smooth tori of solutions of the GNLS equation (4.26) which are periodic in space, quasiperiodic in time, and invariant under phase rotation. For any positive integer *n*, an *n*-torus of GNLS solutions is expressible as

$$A(x,t) = F(\vec{k}x + \vec{\omega}t + \vec{z}), \tag{3.1}$$

where F is a C^∞ function defined over the n-torus $\boldsymbol{T}^n \equiv \boldsymbol{R}^n/\boldsymbol{Z}^n$. Here $\vec{z} \in \boldsymbol{T}^n$ is the toroidal coordinate, $\vec{k} \in \boldsymbol{Z}^n$ is the wavevector of the periodic spatial profile, and $\vec{\omega} \in \boldsymbol{R}^n$ is the frequency vector of the GNLS flow. The invariance under phase rotation means there exists a vector $\vec{l} \in \boldsymbol{Z}^n$ such that

$$F(\vec{l}t_0 + \vec{z}) = \mathrm{e}^{-\mathrm{i}2\pi t_0} F(\vec{z}). \tag{3.2}$$

A torus of the form (3.1) is said to be *nonresonant* if

$$\{\vec{l}t_0 + \vec{k}t_1 + \vec{\omega}t_2 : (t_0, t_1, t_2) \in \boldsymbol{R}^3\} \quad \text{is dense in } \boldsymbol{T}^n. \tag{3.3}$$

Notice that this is a weaker condition than assuming that the components of $\vec{\omega}$ are rationally independent, due to the subtorus swept out by the terms $\vec{l}\,t_0 + \vec{k}t_1$. The torus is otherwise said to be *resonant*, in which case the torus can be foliated into lower-dimensional nonresonant subtori. It is important to note that for the cNLS these solutions exist and can be written in terms of theta functions of several variables [11].

In this context we define the notion of persistence as follows.

Definition 3.1. A nonresonant torus of GNLS solutions of the form (3.1) is said to persist under the GCGL perturbation if and only if there exists an ϵ-dependent family of smooth tori of solutions of the GCGL equation (1.2) of the form

$$A_\epsilon(x,t) = F_\epsilon(\vec{k}x + \vec{\omega}_\epsilon t + \vec{z}), \tag{3.4}$$

each of which satisfies (3.2) with F replaced by F_ϵ, where the frequency vectors $\vec{\omega}_\epsilon$ converge to $\vec{\omega}$ in $\boldsymbol{R}^n$ and the functions F_ϵ converge to F in $C^\infty(\boldsymbol{T}^n)$ as ϵ tends to zero.

Remark 3.1. A family of tori of GCGL solutions associated with a persisting GNLS torus will be nonresonant for a sequence $\{\epsilon_j\}$ that converges to zero in the sense that (3.3) is satisfied by $\vec{l}$, $\vec{k}$, and $\vec{\omega}_{\epsilon_j}$.

We will now use an averaging method to deduce necessary conditions for a nonresonant GNLS torus to persist under the GCGL perturbation. More precisely, we will show that the time average of a particular class of functionals must vanish on the GNLS torus if that torus persists under the GCGL perturbation. This criterion is derived as follows. The time evolution of any functional $\mathcal{F}$ under the GCGL flow is given by (2.11). Because $\mathrm{d}\mathcal{F}(A_\epsilon)/\mathrm{d}t$ is the derivative of a quasiperiodic function, its time average must be zero. Hence, the right side of (2.11) must satisfy

$$0 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathcal{S}^{\mathcal{F}}(A_\epsilon(t))\,\mathrm{d}t. \tag{3.5}$$

Because $\mathcal{S}^{\mathcal{F}}$ is conserved by the flows of phase rotation and spatial translation by (2.15), for every ϵ_j corresponding to one of the nonresonant GCGL tori mentioned in Remark 3.1, this time average may be replaced by a toroidal average using the form (3.4). Hence, (3.5) yields

$$0 = \int_{\boldsymbol{T}^n} \left(\frac{\delta\mathcal{F}}{\delta A^*}\frac{\delta\mathcal{G}}{\delta A} + \frac{\delta\mathcal{F}}{\delta A}\frac{\delta\mathcal{G}}{\delta A^*} \right)(F_{\epsilon_j}(\vec{z}))\,\mathrm{d}^n z. \tag{3.6}$$

Because $\mathcal{F}$ was assumed to be and $\mathcal{G}$ is a smooth function of A over $C^\infty(\boldsymbol{T})$, the above functional will be a smooth function of F_{ϵ_j} over $C^\infty(\boldsymbol{T}^n)$. For example, if $\mathcal{F}$ has the form (2.8) then the above integrand will even be an analytic function of a finite number of derivatives of F_{ϵ_j}. Then because F_{ϵ_j} converges to F in $C^\infty(\boldsymbol{T}^n)$, we can let ϵ_j tend to zero inside the integral in (3.6) and conclude the following.

Proposition 3.1. *A necessary condition for the persistence of a nonresonant GNLS torus in the sense of* Definition 3.1 *is the following. For each functional $\mathcal{F}$ satisfying* (2.14), *the function F that characterizes the GNLS torus through* (3.1) *satisfies the 'Melnikov' selection criterion*

$$0 = \mathcal{M}^{\mathcal{F}}(F) \equiv \int_{\boldsymbol{T}^n} \left(\frac{\delta \mathcal{F}}{\delta A^*} \frac{\delta \mathcal{G}}{\delta A} + \frac{\delta \mathcal{F}}{\delta A} \frac{\delta \mathcal{G}}{\delta A^*} \right) (F(\vec{z}))\, \mathrm{d}^n z, \tag{3.7}$$

or equivalently, any solution A on a persisting GNLS torus must satisfy

$$0 = \mathcal{M}^{\mathcal{F}}(A) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathcal{S}^{\mathcal{F}}(A(t))\, \mathrm{d}t. \tag{3.8}$$

4. Sufficiency and the selection criteria

In this section we show how families of tori of quasiperiodic GNLS solutions can arise within critical sets of linear combinations of conserved functionals. We relate the number of conserved functionals used to generate such tori with the number of independent selection criteria (necessary conditions), showing that the number of these criteria is equal to the number of continuous parameters in the family. We therefore find that generically only isolated tori in the family will satisfy these selection criteria, and that in some simple cases these criteria are sufficient to conclude that a torus persists.

The critical points of a functional $\mathcal{F}$ are those functions B at which its gradient vanishes, $\delta\mathcal{F}/\delta A^*(B) = 0$. The set of all critical points of $\mathcal{F}$ is called the critical set of $\mathcal{F}$. It is a general fact that if a functional $\mathcal{F}$ is conserved by an autonomous flow then its critical sets are invariant under that flow, i.e. its critical sets are also sets of solutions. This can be seen by computing the time derivative of the gradient of $\mathcal{F}$ and using the identity obtained by taking the gradient of $\mathrm{d}\mathcal{F}/\mathrm{d}t = 0$. For the GNLS case one finds

$$\partial_t \frac{\delta \mathcal{F}}{\delta A^*} = -\mathrm{i} \frac{\delta^2 \mathcal{H}}{\delta A^* \delta A} \frac{\delta \mathcal{F}}{\delta A^*} + \mathrm{i} \frac{\delta^2 \mathcal{H}}{\delta A^{*2}} \frac{\delta \mathcal{F}}{\delta A}. \tag{4.1}$$

This equation is linear in $\delta\mathcal{F}/\delta A^*$ and $\delta\mathcal{F}/\delta A$, so if these quantities are initially zero they remain so.

By considering linear combinations of known conserved quantities we can obtain families of critical sets and hence families of solutions. For example, for the GNLS equation, families of critical sets associated with $\mathcal{F} = \mathcal{J} - \alpha\mathcal{N}$ for some real α satisfy

$$\frac{\delta \mathcal{J}}{\delta A^*}(B) = \alpha \frac{\delta \mathcal{N}}{\delta A^*}(B), \tag{4.2}$$

while those associated with $\mathcal{F} = \mathcal{H} - \beta\mathcal{J} - \alpha\mathcal{N}$ for some real α and β satisfy

$$\frac{\delta \mathcal{H}}{\delta A^*}(B) = \beta \frac{\delta \mathcal{J}}{\delta A^*}(B) + \alpha \frac{\delta \mathcal{N}}{\delta A^*}(B). \tag{4.3}$$

These cases exhaust the nontrivial families of critical sets associated with linear combinations of $\mathcal{N}$, $\mathcal{J}$, and $\mathcal{H}$. As we will see, Eqs. (4.2) and (4.3) give rise to families of rotating wave and traveling wave solutions, respectively.

4.1. Rotating waves

By writing out (4.2) explicitly, we see that the critical points are those B in $H^1(\boldsymbol{T})$ that satisfy

$$-\mathrm{i}\partial_x B = \alpha B. \tag{4.4}$$

The critical points thereby have the form $B_k(x) = a\,e^{i2\pi kx}$ and $\alpha = 2\pi k$, where k is an integer and a is a complex constant. The GNLS solutions lying within these critical sets are the so-called rotating wave solutions, which for each integer k have the form

$$A_k(x,t) = B_k(x)\,e^{-i\omega_k t} = a\exp(i(2\pi kx - \omega_k t)), \tag{4.5}$$

where $\omega_k = 4\pi^2k^2 + h'(|a|^2)$. These are of the form (3.1) with $n = 1$, where the toroidal (or angle) coordinate is $z = \arg(a)/2\pi$. For each k the level sets of $\mathcal{N}$ are fixed by $|a|$ and therefore foliate the family (4.5) into one-tori (circles), each of whose angle is identified with the flow of phase rotation (2.7). Thus we have a two-parameter family of one-tori parameterized by $|a|$ and k, where the latter parameter is quantized by the periodicity condition.

For these families of solutions the necessary criteria given in Proposition 3.1 are sufficient to assert persistence. In fact, persistence holds when the sole necessary Melnikov criterion (3.8) with $\mathcal{F}$ set equal to $\mathcal{N}$ holds. Indeed, this criterion yields

$$0 = \mathcal{M}^{\mathcal{N}}(A_k) = \mathcal{S}^{\mathcal{N}}(B_k) = 2\int_0^1 (|\partial_x B_k|^2 + g'(|B_k|^2)|B_k|^2)\,dx = 2(4\pi^2k^2 + g'(|a|^2))|a|^2. \tag{4.6}$$

This is clearly sufficient because when $g'(|a|^2) = -4\pi^2k^2$ the GNLS rotating wave (4.5) is also an exact solution of the GCGL equation.

Remark 4.1. Note that the (in this case sufficient) Melnikov criteria for persistence was obtained using only the functional $\mathcal{N}$ whose level sets foliate the family of rotating waves. Also note that the Melnikov criteria selected isolated values of $\mathcal{N}$. This reflects a general pattern to which the following examples also conform. Namely the number of real continuous parameters of a family of tori equals the number of its foliating functionals and also equals the number of Melnikov criteria obtained from these functionals. This equality means that the Melnikov criteria generically select isolated parameter values.

4.2. Traveling waves

By writing out (4.3) explicitly, we see that the critical points are those B in $H^1(\boldsymbol{T})$ that satisfy

$$\partial_{xx}B + \alpha B - i\beta\partial_x B - h'(|B|^2)B = 0. \tag{4.7}$$

This is a second-order complex-valued ODE, which therefore has four degrees of freedom in addition to the parameters α and β. The fact that $\mathcal{N}$ and $\mathcal{J}$ Poisson commute with $\mathcal{F} = \mathcal{H} - \beta\mathcal{J} - \alpha\mathcal{N}$ implies the integrands of $\{\mathcal{F}, \mathcal{N}\}$ and $\{\mathcal{F}, \mathcal{J}\}$ are each a perfect derivative in x, any primitive of which is a real integral of (4.7). There will therefore be two spatial periodicity conditions that involve these integrals and the α and β.

When the gradients of $\mathcal{N}$ and $\mathcal{J}$ are linearly independent at each point of these critical sets, and when these critical sets are compact, the level sets of $\mathcal{N}$ and $\mathcal{J}$ will foliate them into 2-tori whose angles are given by the flows of phase rotation and spatial translation (2.7). This family of 2-tori will therefore be parameterized by four parameters, two of which will be continuous and two of which will be quantized by the spatial periodicity conditions. If we denote the two continuous parameters by η and ρ, then it is clear from (2.1), (2.7) and (4.3) that the GNLS solutions in these critical sets are so-called traveling waves of the form

$$A(x,t) = e^{-i\alpha t}B(x - \beta t; \eta, \rho), \tag{4.8}$$

where α and β are given by smooth functions of η and ρ as

$$\alpha = a(\eta, \rho), \qquad \beta = b(\eta, \rho). \tag{4.9}$$

These solutions are of the form (3.1) with $n = 2$, where $\vec{z} = (z_0, z_1) \in \boldsymbol{T}^2$, $F(\vec{z}) = e^{i2\pi z_0}B(z_1)$, $\vec{k} = (0, 1)$, and $\vec{\omega} = (\alpha/2\pi, \beta)$. Moreover, from (3.2) one sees that $\vec{l} = (-1, 0)$, whereby this torus always satisfies the nonresonance condition (3.3). Note that the toroidal coordinates are proportional to the times associated with the flows of phase rotation and spatial translation (2.7).

For every functional $\mathcal{F}$ that Poisson commutes with $\mathcal{H}$, $\mathcal{J}$, and $\mathcal{N}$, the Melnikov selection criterion (3.8) applied to traveling waves (4.8) reduces to

$$0 = \mathcal{M}^{\mathcal{F}}(A) = \mathcal{S}^{\mathcal{F}}(B). \tag{4.10}$$

With $\mathcal{F}$ set equal to $\mathcal{N}$ and $\mathcal{J}$, this yields the selection criteria

$$0 = \mathcal{S}^{\mathcal{N}}(B(\eta, \rho)), \qquad 0 = \mathcal{S}^{\mathcal{J}}(B(\eta, \rho)), \tag{4.11}$$

where by the definition (2.12) of $\mathcal{S}^{\mathcal{F}}$ one has

$$\mathcal{S}^{\mathcal{N}}(B) = 2\int_0^1 (|\partial_x B|^2 + g'(|B|^2)|B|^2)\,\mathrm{d}x, \tag{4.12}$$

$$\mathcal{S}^{\mathcal{J}}(B) = -\mathrm{i}\int_0^1 ((\partial_x B^* \partial_{xx} B - \partial_x B \partial_{xx} B^*) + g'(|B|^2)(B^* \partial_x B - B \partial_x B^*))\,\mathrm{d}x. \tag{4.13}$$

It might be expected that the two criteria (4.11) will select isolated values of two variables η and ρ. This will certainly be the case if, in addition to satisfying (4.11), η and ρ also satisfy the transversality condition

$$\det\begin{pmatrix} \partial_\eta \mathcal{S}^{\mathcal{N}}(B(\eta, \rho)) & \partial_\rho \mathcal{S}^{\mathcal{N}}(B(\eta, \rho)) \\ \partial_\eta \mathcal{S}^{\mathcal{J}}(B(\eta, \rho)) & \partial_\rho \mathcal{S}^{\mathcal{J}}(B(\eta, \rho)) \end{pmatrix} \neq 0. \tag{4.14}$$

Indeed, in [2] we show that if the selection criteria (4.11) are complemented by this transversality condition then the traveling waves corresponding to these selected isolated values do in fact persist. Note that as in Remark 4.1, the number of continuous real parameters for the traveling wave families, the number of functionals that foliate these families, and the number of necessary Melnikov criteria are all equal (to two in this case), leading generically to the persistence of only isolated traveling waves.

The selection criteria (4.11) can be seen to also arise directly from the profile equation obtained by seeking GCGL solutions of the traveling waveform

$$A_\epsilon(x, t) = \mathrm{e}^{-\mathrm{i}\alpha_\epsilon t} B_\epsilon(x - \beta_\epsilon t), \tag{4.15}$$

which is of the form (3.4) with $n = 2$. The profile equation so obtained is

$$\frac{\delta\mathcal{H}}{\delta A^*}(B_\epsilon) - \beta_\epsilon \frac{\delta\mathcal{J}}{\delta A^*}(B_\epsilon) - \alpha_\epsilon \frac{\delta\mathcal{N}}{\delta A^*}(B_\epsilon) = \epsilon\mathrm{i}\frac{\delta\mathcal{G}}{\delta A^*}(B_\epsilon). \tag{4.16}$$

Then for every functional $\mathcal{F}$ that Poisson commutes with $\mathcal{H}$, $\mathcal{J}$, and $\mathcal{N}$, a direct calculation shows that B_ϵ must satisfy

$$0 = \mathcal{S}^{\mathcal{F}}(B_\epsilon), \tag{4.17}$$

where $\mathcal{S}^{\mathcal{F}}$ is defined by (2.12). If this holds with $\mathcal{F} = \mathcal{N}$ and $\mathcal{F} = \mathcal{J}$, then by (4.16) it holds for $\mathcal{F} = \mathcal{H}$. Persistence of B requires that $B_\epsilon \to B$ in $C^\infty(\boldsymbol{T})$, so passing to the $\epsilon \to 0$ limit with $\mathcal{F} = \mathcal{N}$ and $\mathcal{F} = \mathcal{J}$ then recovers (4.11).

The sufficiency of the selection criteria (4.11) and the transversality condition (4.14) is established in [2] by showing that the profile equation (4.16) has solutions B_ϵ in $H^1(\boldsymbol{T})$. Evidence for this is provided by the fact that (4.16) can be solved by a formal expansion whenever $(\eta, \rho) = (\eta^{(0)}, \rho^{(0)})$ satisfies (4.11) and (4.14). Specifically, one seeks a solution of the form (4.15) with

$$\alpha_\epsilon = a(\eta_\epsilon, \rho_\epsilon), \qquad \beta_\epsilon = b(\eta_\epsilon, \rho_\epsilon), \tag{4.18}$$

where a and b are the same smooth functions as in (4.9), and with B_ϵ, η_ϵ and ρ_ϵ expanded as

$$B_\epsilon(x) = B(x; \eta^{(0)}, \rho^{(0)}) + \epsilon B^{(1)}(x) + \epsilon^2 B^{(2)}(x) + \cdots, \qquad \eta_\epsilon = \eta^{(0)} + \epsilon\eta^{(1)} + \epsilon^2\eta^{(2)} + \cdots,$$
$$\rho_\epsilon = \rho^{(0)} + \epsilon\rho^{(1)} + \epsilon^2\rho^{(2)} + \cdots. \tag{4.19}$$

Upon substituting expansions (4.19) into (4.17), one finds that two solvability conditions for $B^{(k)}$ arise at order ϵ^k for each $k \geq 1$. The selection criteria (4.11) ensure the two solvability conditions for $B^{(1)}$ are satisfied. The transversality condition (4.14) ensures that $\eta^{(k)}$ and $\rho^{(k)}$ can be determined for each $k \geq 1$ in such a way that the two solvability conditions for $B^{(k+1)}$ are satisfied.

4.3. More general quasiperiodic solutions

To construct more general families of quasiperiodic solutions by the above method we need additional conserved functionals beyond $\mathcal{N}$, $\mathcal{J}$, and $\mathcal{H}$. With the notable exception of the cNLS case, we know of no example where such additional functionals are available. Nevertheless, we proceed in the abstract setting of (2.1) where $\mathcal{H}$ is not assumed to be given by (2.2) but is assumed to satisfy (2.6).

Fix an integer $m > 2$. Suppose $\{\mathcal{F}_j\}_{j=0}^{m}$ is any set of functionals with $\mathcal{F}_0 = \mathcal{N}$, $\mathcal{F}_1 = \mathcal{J}$, and $\mathcal{F}_2 = \mathcal{H}$, that mutually Poisson commute, and that has linearly independent gradients. Let $\{\alpha_j\}_{j=0}^{m-1}$ be m real parameters. Consider the family of critical sets defined by

$$\frac{\delta\mathcal{F}_\alpha}{\delta A^*}(B) = 0, \quad \text{where } \mathcal{F}_\alpha = (\mathcal{F}_m + \alpha_{m-1}\mathcal{F}_{m-1} + \cdots + \alpha_0\mathcal{F}_0). \tag{4.20}$$

If this is an kth-order complex-valued ODE, its general solution has $2k$ degrees of freedom in addition to the m parameters $\{\alpha_j\}$. It will generally have m integrals that arise because the fact $\{\mathcal{F}_\alpha, \mathcal{F}_j\} = 0$ for $j = 0, 1, \ldots, m-1$ implies the integrand of each $\{\mathcal{F}_\alpha, \mathcal{F}_j\}$ is a perfect derivative in x, any primitive of which is an integral. If we suppose these exhaust the integrals, there will then be $2k - m$ spatial periodicity conditions that involve these integrals and the $\{\alpha_j\}$.

When the gradients of the $\{\mathcal{F}_j\}_{j=0}^{m-1}$ are linearly independent at each point of a critical set, the level sets of $\{\mathcal{F}_j\}$ will foliate that critical set into m-dimensional manifolds with local coordinates provided by the times $\{t_j\}_{j=0}^{m-1}$ associated with the flows

$$\partial_{t_j} A = -\mathrm{i}\frac{\delta\mathcal{F}_j}{\delta A^*}. \tag{4.21}$$

If we suppose moreover that these critical sets are compact, as is always the case for the defocusing cNLS and for most cases for the focusing cNLS, then each connected component of these m-dimensional manifolds (being a compact, connected, Abelian Lie group) is an m-torus. It then has a representation of the form (3.1) with $n = m$. We therefore expect to have families of m-tori parameterized by m real continuous parameters and $2k - m$ quantized parameters.

Suppose the $2k - m$ quantized parameters have been fixed, and a particular collection of nonresonant m-tori with these parameters and m remaining real continuous parameters has been singled out. The nonresonance condition ensures that when evaluated on an m-torus each of the necessary Melnikov criteria (3.7) will integrate out the m angle parameters, leaving just the m continuous parameters. But there are m such criteria, one corresponding to (3.7) with $\mathcal{F}$ set equal to each $\mathcal{F}_j$ for $j = 0, 1, \ldots, m-1$. Because there are m continuous real parameters for such a family, these m conditions generically are expected to select isolated values of these parameters, confirming the generality of Remark 4.1 for necessary criteria.

The foregoing examples of rotating waves and traveling waves were examples of this general pattern with $m = 1$ and $m = 2$, respectively. In those examples we also gave sufficient criteria. However, due to the expectation that the necessary criteria will generally select only isolated tori, it is clear that the primary obstruction to persistence are the necessary criteria. In the case of traveling waves, for example, waves which meet the necessary criteria generically persist.

As we said above, the cNLS equation (1.4) is one case for which there are an infinite number of conserved functionals $\{\mathcal{F}_j\}_{j=0}^{\infty}$ that mutually Poisson commute, and that have linearly independent gradients [10,11]. These can be ordered so that $\mathcal{F}_0 = \mathcal{N}$, $\mathcal{F}_1 = \mathcal{J}$, $\mathcal{F}_2 = \mathcal{H}$, and $\delta\mathcal{F}_j/\delta A^*$ is a jth-order, complex-valued, nonlinear differential operator. Therefore in this case $k = m$ when the preceding discussion is applied to the finite set $\{\mathcal{F}_j\}_{j=0}^{m}$. The associated necessary Melnikov criteria have been expressed in terms of the machinery of the associated inverse spectral transform. More specifically, they are seen to have an elegant formulation in terms of the Floquet discriminant of the associated Zakharov–Shabat operator. They will be analyzed in [4].

4.4. The Lyapunov case

A special case for which a fairly complete picture emerges is the so-called Lyapunov case of the GCGL equation. This arises for any flow of the abstract form (2.9) when $\mathcal{G}$ itself Poisson commutes with $\mathcal{H}$. In that case, when equation (2.11) is applied to determine the evolution of $\mathcal{G}$, it reduces to

$$\frac{\mathrm{d}\mathcal{G}}{\mathrm{d}t} = -\epsilon 2 \int_0^1 \left| \frac{\delta\mathcal{G}}{\delta A^*} \right|^2 \mathrm{d}x. \tag{4.22}$$

Because the right side is always nonpositive, $\mathcal{G}$ is nonincreasing on all solutions of the GCGL equation and may be used formally like a Lyapunov functional. From Proposition 3.1, one obtains

$$\mathcal{M}^{\mathcal{G}}(A) = 0 \iff \frac{\delta\mathcal{G}}{\delta A^*}(A) = 0. \tag{4.23}$$

Therefore, the only solutions that are selected are those in the critical set of $\mathcal{G}$. Indeed, the GNLS solutions in this critical set are exact Lyapunov GCGL solutions. The selection criteria is therefore sufficient to determine the persistence of solutions. When this observation is applied in the cNLS setting, it allows one to construct flows of the form (2.9) whose attractor contains the critical set of any conserved functional $\mathcal{G}$.

Specifically, when $\mathcal{H}$ and $\mathcal{G}$ have the forms (2.2) and (2.10), one finds that in the Lyapunov case the functional $\mathcal{G}$ must take the form

$$\mathcal{G} = \mathcal{H} - r\mathcal{N}. \tag{4.24}$$

In this case equation the GCGL equation takes the form

$$\partial_t A = \frac{\delta\mathcal{N}}{\delta A^*} - (\mathrm{i} + \epsilon)\frac{\delta\mathcal{H}}{\delta A^*}, \tag{4.25}$$

or what is the same, using (2.2)

$$\partial_t A = rA + (\mathrm{i} + \epsilon)\partial_{xx} A - (\mathrm{i} + \epsilon)h'(|A|^2)A. \tag{4.26}$$

The selected solutions thereby have the separable form

$$A(x, t) = \mathrm{e}^{-\mathrm{i}rt} B(x),$$

where B is a critical point of $\mathcal{G}$. A detailed study of the cubic Lyapunov case $h(\xi) = \xi^2$ has been carried out by Horsch and Levermore [6], where the dynamical stability of the selected solutions is analyzed.

We remark that the only cases in which the selected GNLS solution is an exact GCGL solution are the general rotating waves and the Lyapunov cases treated above. In all other cases, GNLS solutions that persist must deform under the GCGL perturbation, as in the traveling wave case analyzed in [2,3].

5. Necessary criteria for persistence of homoclinics

In this section, we present the equivalent criteria to those derived in Section 3 for the persistence of temporally homoclinic solutions of the GNLS equation under a GCGL perturbation.

We begin by first discussing the existence of GNLS homoclinic solutions. For the focusing cNLS equation (1.4) (with the plus sign) it is known that rotating waves unstable with respect to the sideband perturbations with wavenumber $k_m = 2\pi m$ are homoclinic points of temporally homoclinic solutions [27,28]. In the general case, for which the GNLS is not integrable, it is less clear as to when there are homoclinic solutions. In some instances the GNLS may support many homoclinics, in others none at all. In any case, we now develop a general criteria for their persistence that may be applied whenever they do exist.

Let $A(x, t)$ be a solution of the GNLS equation which is homoclinic to a torus of traveling waves of the form (4.8). This means that there exist constants $\tau^{\pm}$ and $z^{\pm}$ such that for every nonnegative integer j one has

$$\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}\alpha t}\partial_x^j A(x+\beta t, t) = \mathrm{e}^{i\tau^{\pm}}\partial_x^j B(x - z^{\pm}), \tag{5.1}$$

where the limit is uniform over x. Clearly, a prerequisite for $A(x, t)$ to persist under the GCGL perturbation is for the torus of traveling waves containing $B(x)$ to persist. That this is the case requires by (2.12) that $\mathcal{S}^{\mathcal{F}}(B) = 0$ for every functional $\mathcal{F}$ that Poisson commutes with $\mathcal{N}$, $\mathcal{J}$, and $\mathcal{H}$. Let $B_\epsilon(x)$ be the GCGL traveling wave profiles that then converge to $B(x)$. By (4.17) these also satisfy $\mathcal{S}^{\mathcal{F}}(B_\epsilon) = 0$ for every functional $\mathcal{F}$ that Poisson commutes with $\mathcal{N}$, $\mathcal{J}$, and $\mathcal{H}$.

Definition 5.1. A GNLS solution $A(x, t)$ which is homoclinic to a torus of traveling waves is said to persist under the GCGL perturbation if and only if there exists an ϵ-dependent family $A_\epsilon(x, t)$ of GCGL solutions that satisfies two requirements. First, that for each ϵ it is homoclinic to the torus of GCGL traveling waves containing $B_\epsilon(x)$ in the sense that

$$\lim_{t\to\pm\infty} \mathcal{F}(A_\epsilon(t)) = \mathcal{F}(B_\epsilon), \tag{5.2}$$

$$\lim_{t\to\pm\infty} \mathcal{S}^{\mathcal{F}}(A_\epsilon(t)) = \mathcal{S}^{\mathcal{F}}(B_\epsilon) = 0 \tag{5.3}$$

for every functional $\mathcal{F}$ that Poisson commutes with $\mathcal{N}$, $\mathcal{J}$, and $\mathcal{H}$. Second, that it deforms to the GNLS homoclinic orbit $A(x, t)$ as $\epsilon \to 0$ in the sense that for every nonnegative integer j one has

$$\lim_{\epsilon\to 0} \partial_x^j A_\epsilon(x, t) = \partial_x^j A(x, t), \tag{5.4}$$

where the limit is uniform in (x, t) over compact subsets of $\boldsymbol{T} \times \boldsymbol{R}$, and that for every functional $\mathcal{F}$ that Poisson commutes with $\mathcal{N}$, $\mathcal{J}$, and $\mathcal{H}$ one has

$$\lim_{\epsilon\to 0} \mathcal{S}^{\mathcal{F}}(A_\epsilon(t)) = \mathcal{S}^{\mathcal{F}}(A(t)), \tag{5.5}$$

where the limit is in $L^1(\boldsymbol{R})$ over t.

Eq. (2.11) for the evolution of any GNLS conserved functional $\mathcal{F}$ evaluated at $A_\epsilon(x, t)$ may be re-expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{F}(A_\epsilon(t)) - \mathcal{F}(B_\epsilon)) = -\epsilon\mathcal{S}^{\mathcal{F}}(A_\epsilon(t)).$$

By (5.2) the integration of this equation gives

$$0 = -\frac{1}{\epsilon}(\mathcal{F}(A_\epsilon(t)) - \mathcal{F}(B_\epsilon))\Big|_{-\infty}^{+\infty} = \int_{-\infty}^{+\infty} \mathcal{S}^{\mathcal{F}}(A_\epsilon(t))\,\mathrm{d}t. \tag{5.6}$$

Because $\mathcal{S}^{\mathcal{F}}(A_\epsilon(t)) \to \mathcal{S}^{\mathcal{F}}(A(t))$ as $\epsilon \to 0$ in $L^1(\boldsymbol{R})$ over t by (5.5), passing to the limit in (5.6) gives the following.

Proposition 5.1. *A necessary condition for the persistence in the sense of Definition 5.1 of a GNLS solution A that is homoclinic to a persistent torus of GNLS traveling waves is that it satisfy the 'Melnikov' selection criterion*

$$0 = \mathcal{M}_H^{\mathcal{F}}(A) \equiv \int_{-\infty}^{+\infty} \mathcal{S}^{\mathcal{F}}(A(t))\,\mathrm{d}t \tag{5.7}$$

for every functional $\mathcal{F}$ that satisfies (2.14).

6. Nonpersistence of homoclinics to rotating waves

In this section, we use the criteria (5.7) to prove the following proposition.

Proposition 6.1. *Every solution of the GNLS equation* (4.26) *that is homoclinic to a persistent circle of GNLS rotating waves does not persist in the sense of Definition 5.1 when the GCGL perturbation satisfies*

$$\xi \mapsto \xi g'(\xi) - h(\xi) \quad \textit{is strictly convex over } \boldsymbol{R}_+. \tag{6.1}$$

Remark 6.1. Proposition 6.1 tells us that all such homoclinic solutions that might exist for the GNLS equation with the focusing power law form for $h(\xi)$ given in (1.3) do not persist under the class of GCGL perturbations for which

$$\xi \mapsto \xi g'(\xi) + \frac{2}{s}\xi^s \quad \text{is strictly convex over } \boldsymbol{R}_+. \tag{6.2}$$

This condition includes the general power law form for $g(\xi)$ given in (1.3).

Remark 6.2. In particular, the known homoclinic solutions of the focusing cNLS equation that terminate at rotating waves [27,28] are destroyed by the class of GCGL perturbations satisfying (6.2) with $s = 2$. This result contrasts with that of McLaughlin et al. [25], who prove the persistence of a focusing cNLS homoclinic for a perturbation that breaks the phase rotation symmetry. This symmetry breaking means that there is only one such homoclinic that persists, not a circle of them. Indeed, their cNLS homoclinic persists in the sense of Definition 5.1, but they show much more. They show the perturbed homoclinic exhibits a very slow dynamics along the circle of rotating waves for very large times that ultimately returns it to the exact same rotating wave from which it departed. In other words, they show that a cNLS orbit which connects two points on a circle of rotating waves is approximated by perturbed orbits that are homoclinic to one of the two points. As ϵ tends to zero, the dynamics along the circle takes place at later times and happens even slower once it begins. This lack of uniformity is consistent with (5.4).

To prove Proposition 6.1, it is enough to show that the necessary condition cannot hold for a specific functional $\mathcal{F}$. We do this for the mass functional $\mathcal{N}$ given by (2.5). The associated selection criterion (5.7) becomes

$$0 = \mathcal{M}_H^{\mathcal{N}}(A) = \int_{-\infty}^{+\infty} \mathcal{S}^{\mathcal{N}}(A)\,\mathrm{d}t, \tag{6.3}$$

where A is a candidate GNLS homoclinic solution and

$$\mathcal{S}^{\mathcal{N}}(A) = \int_0^1 (|\partial_x A|^2 + g'(|A|^2)|A|^2)\,\mathrm{d}x. \tag{6.4}$$

A prerequisite for the integral in (6.3) to vanish is that the rotating wave at which this homoclinic solution terminates also persists. Otherwise, the integrand would be finite as $t \to \pm\infty$, and the integral would diverge. The condition (4.6) for rotating wave persistence yields

$$0 = 4\pi^2 n^2 + g'(|a|^2). \tag{6.5}$$

Observe that we can write the functional (6.4) that appears in the selection criterion (6.3) in terms of the Hamiltonian $\mathcal{H}$ as

$$\mathcal{S}^{\mathcal{N}}(A) = \mathcal{H}(A) + \int_0^1 (g'(|A|^2)|A|^2 - h(|A|^2))\,\mathrm{d}x. \tag{6.6}$$

By the strict convexity (6.1), Jensen's inequality gives

$$\int_0^1 (g'(|A|^2)|A|^2 - h(|A|^2))\,\mathrm{d}x \geq g'(\mathcal{N}(A))\mathcal{N}(A) - h(\mathcal{N}(A)), \tag{6.7}$$

where the equality holds only if $|A|$ is constant in x. Because $\mathcal{H}$ and $\mathcal{N}$ are conserved quantities, they can be evaluated at the end points of the homoclinic, i.e., at the rotating wave, to yield

$$\mathcal{H}(A) = 4\pi^2 n^2 |a|^2 + h(|a|^2), \qquad \mathcal{N}(A) = |a|^2. \tag{6.8}$$

Now combine (6.6)–(6.8) while using the fact that for all homoclinic solutions $|A|$ is not constant in x (so that Jensen's inequality (6.7) is strict), and evaluate the resulting lower bound with (6.5) to obtain the inequality

$$\mathcal{S}^{\mathcal{N}}(A) > 4\pi^2 n^2 |a|^2 + g'(|a|^2)|a|^2 = 0. \tag{6.9}$$

This contradicts (6.3). Therefore no homoclinic solution of the GNLS equation (4.26) terminating at a rotating wave can persist under the GCGL perturbation generated by a $g(\xi)$ that satisfies (6.1).

Remark 6.3. If one tries to apply an analogous argument to show the nonpersistence of homoclinic solutions that terminate at traveling waves of the form (4.8), it yields the bound

$$\mathcal{S}^{\mathcal{N}}(A) > \mathcal{H}(B) + g'(\mathcal{N}(B))\mathcal{N}(B) - h(\mathcal{N}(B)). \tag{6.10}$$

But the expression on the right does not combine to become the traveling wave selection criterion (4.12) the way the analogous expression in (6.9) became the rotating wave selection criterion (6.5). Rather, by applying (6.6) and (6.7) with A replaced by B, one sees that (4.12) implies

$$\mathcal{H}(B) + g'(\mathcal{N}(B))\mathcal{N}(B) - h(\mathcal{N}(B)) < \mathcal{S}^{\mathcal{N}}(B) = 0.$$

So the lower bound obtained for $\mathcal{M}_H^{\mathcal{N}}$ in (6.3) from (6.10) is $-\infty$.

Analysis of the persistence of GNLS homoclinics does not, however, tell us whether or not *new* homoclinic solutions are created by the perturbation. New homoclinics might be created, for example, as we will show, if the parameters of the perturbation are chosen near values that cause a nonhyperbolic rotating wave to persist. A nonhyperbolic rotating wave by definition possesses a pair of zero eigenvalues with respect to its linear stability in the GNLS equation under sideband perturbations. For example, as occurs for the focusing cNLS equation, nonhyperbolic rotating waves must exist at the boundaries of families of solutions homoclinic to rotating waves, where the eigenvalues associated with the modulational instability generating the homoclinics in the family pass through the origin and the modulational instability changes to neutral stability. The fact that these points are nonhyperbolic suggests that some delicate and complicated structure, including homoclinic solutions, might be created near these points under the GCGL perturbation.

We now present a numerical example of a case where a new homoclinic solution seems to be created at a nonhyperbolic rotating wave. Of course, this new homoclinic cannot be a C^∞ deformation of a GNLS homoclinic as shown above. For the focusing cNLS equation under spatially periodic boundary conditions with unit period, the $n = 0$ rotating waves with $a > \pi$ form the endpoints of homoclinic solutions generated by sideband instabilities of wavenumber $k_1 = 2\pi$ (there are additional homoclinics as well at higher amplitudes for other wavenumbers. See [27,28] for derivations, explicit expressions, etc.). At $a = \pi$ there exists a nonhyperbolic rotating wave on the

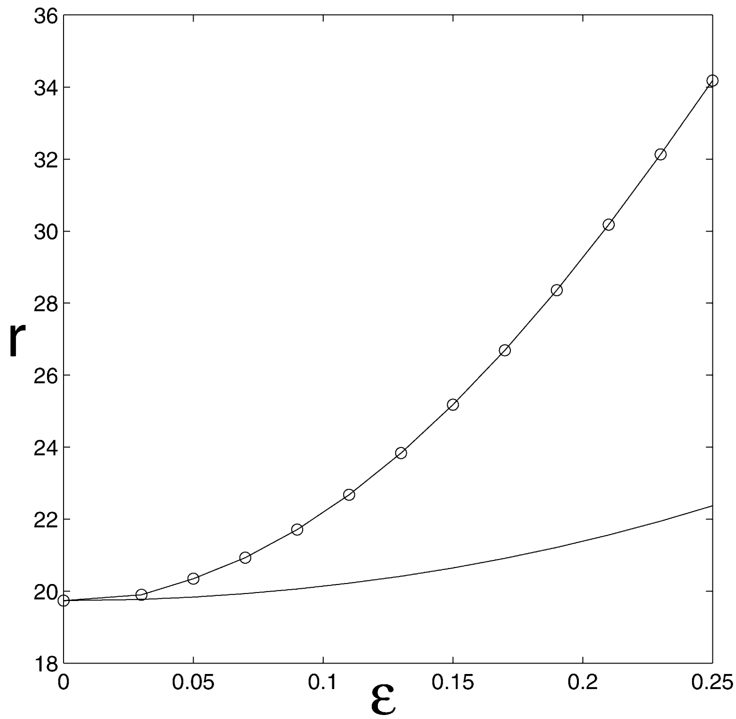

Fig. 1. Numerical evidence for the creation of a homoclinic solution out of the rotating wave with $a = \pi$. The solid line with circles is the parameter curve $r(\epsilon)$ at which a homoclinic solution of the focusing cCGL equation was found numerically. The dashed curve shows the neutral stability curve of the $n = 0$ rotating wave, along which the wave possesses a pair of zero eigenvalues. The two curves converge as $\epsilon \to 0$, suggesting that the homoclinic solution converges to the rotating wave with $a = \pi$.

boundary of the family. We consider the dynamics of the focusing cCGL equation (Eq. (1.5) with minus sign) for small values of ϵ and $q = 1/2$, with values of r close to those for which the nonhyperbolic rotating wave persists.

Numerically, we can locate a homoclinic solution of the focusing cCGL equation, which exists along a one-dimensional parameter curve $r(\epsilon)$ which converges as $\epsilon \to 0$ to the value of r which causes the rotating wave at $a = \pi$ to persist. As $\epsilon \to 0$, the homoclinic orbit shrinks to zero in size, collapsing into the nonhyperbolic rotating wave, i.e. the amplitude of the homoclinic shrinks to a constant in x and therefere it is not a C^{∞} deformation of a cNLS homoclinic. Fig. 1 shows the numerically obtained curve, which was found by a simple ‘shooting’ procedure to find the homoclinic. In the figure, the convergence of the focusing cCGL homoclinic to the $a = \pi$ rotating wave is implied by the convergence of $r(\epsilon)$ (solid line with circles) to the neutral stability curve of the $n = 0$ rotating wave (dashed line), which intersects the nonhyperbolic focusing cNLS rotating wave at $a = \pi$.

Remark 6.4. This family of codimension 1 homoclinic solutions in the focusing cCGL equation is slightly different structurally from the focusing cNLS homoclinics associated with the $n = 0$ rotating wave, which were destroyed by the focusing cCGL perturbation. Whereas the focusing cNLS homoclinics leave and return to the focusing cNLS rotating waves along directions associated with modulational instability in the $k_1 = 2\pi$ wavenumber direction, the focusing cCGL homoclinics leave along a k_1 direction but return along the $k_0 = 0$ direction, the direction of pure amplitude perturbations of the rotating wave. The latter direction becomes a hyperbolic (stable) direction for the rotating wave under the focusing cCGL perturbation. This direction of return is in fact what one would expect generically because a homoclinic solution usually returns along the weakest stable eigenvector.

Remark 6.5. The family of numerical focusing cCGL homoclinic solutions found here also appears as the first homoclinic orbit located as the driving parameter is increased in the study by Luce [21] and also is related to a homoclinic solution of the Lorenz equations, which were shown to govern the dynamics near the $a = \pi$ rotating wave by Malomed and Nepomnyashchy [29].

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