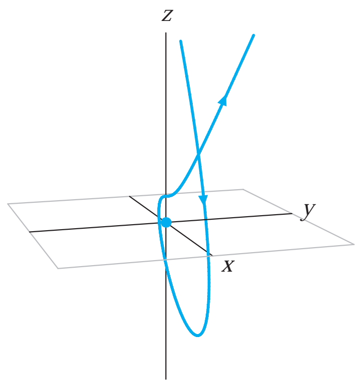

Figure 3.65
Phase space for system $d\mathbf{Y}/dt = \mathbf{AY}$.

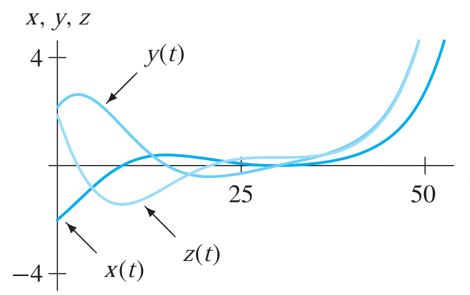

Figure 3.66
Graphs of $x(t)$, $y(t)$ and $z(t)$ for the indicated solution in Figure 3.65.

solution shown, all three coordinates tend to infinity as t increases because the eigenvector for the eigenvalue λ_1 has nonzero components for all three variables.

Three linearly independent solutions of this system are given in the first example of this section (see page 361). We can see from this example that linear systems in three dimensions can be quite complicated (even when many of the coefficients are zero). However, the qualitative behavior is still determined by the eigenvalues, so it is possible to classify these systems without completely solving them.

EXERCISES FOR SECTION 3.8

1. Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY} = \begin{pmatrix} 0 & 0.1 & 0 \\ 0 & 0 & 0.2 \\ 0.4 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Check that the functions

$$\mathbf{Y}_2(t) = e^{-0.1t} \begin{pmatrix} -\cos\left(\sqrt{0.03}\,t\right) - \sqrt{3}\sin\left(\sqrt{0.03}\,t\right) \\ -2\cos\left(\sqrt{0.03}\,t\right) + 2\sqrt{3}\sin\left(\sqrt{0.03}\,t\right) \\ 4\cos\left(\sqrt{0.03}\,t\right) \end{pmatrix}$$

and

$$\mathbf{Y}_3(t) = e^{-0.1t} \begin{pmatrix} -\sin\left(\sqrt{0.03}\,t\right) + \sqrt{3}\cos\left(\sqrt{0.03}\,t\right) \\ -2\sin\left(\sqrt{0.03}\,t\right) - 2\sqrt{3}\cos\left(\sqrt{0.03}\,t\right) \\ 4\sin\left(\sqrt{0.03}\,t\right) \end{pmatrix}$$

are solutions to the system.

2. If a vector $\mathbf{Y}_3$ lies in the plane determined by the two vectors $\mathbf{Y}_1$ and $\mathbf{Y}_2$, then we can write $\mathbf{Y}_3$ as a linear combination of $\mathbf{Y}_1$ and $\mathbf{Y}_2$. That is,

$$\mathbf{Y}_3 = k_1\mathbf{Y}_1 + k_2\mathbf{Y}_2$$

for some constants k_1 and k_2. But then

$$k_1\mathbf{Y}_1 + k_2\mathbf{Y}_2 - \mathbf{Y}_3 = (0, 0, 0).$$

Show that if

$$k_1\mathbf{Y}_1 + k_2\mathbf{Y}_2 + k_3\mathbf{Y}_3 = (0, 0, 0),$$

with not all of k_1, k_2, and $k_3 = 0$, then the vectors are not linearly independent. [*Hint*: Start by assuming that $k_3 \neq 0$ and show that $\mathbf{Y}_3$ is in the plane determined by $\mathbf{Y}_1$ and $\mathbf{Y}_2$. Then treat the other cases.] Note that this computation leads to the theorem that three vectors $\mathbf{Y}_1$, $\mathbf{Y}_2$, and $\mathbf{Y}_3$ are linearly independent if and only if the only solution of

$$k_1\mathbf{Y}_1 + k_2\mathbf{Y}_2 + k_3\mathbf{Y}_3 = (0, 0, 0)$$

is $k_1 = k_2 = k_3 = 0$.

3. Using the technique of Exercise 2, determine whether or not the following sets of three vectors are linearly independent.

(a) $(1, 2, 1)$, $(1, 3, 1)$, $(1, 4, 1)$
(b) $(2, 0, -1)$, $(3, 2, 2)$, $(1, -2, -3)$
(c) $(1, 2, 0)$, $(0, 1, 2)$, $(2, 0, 1)$
(d) $(-3, \pi, 1)$, $(0, 1, 0)$, $(-2, -2, -2)$

In Exercises 4–7, consider the linear system $d\mathbf{Y}/dt = \mathbf{AY}$ with the coefficient matrix $\mathbf{A}$ specified. Each of these systems decouples into a two-dimensional system and a one-dimensional system. For each exercise,

(a) compute the eigenvalues,
(b) determine how the system decouples,
(c) sketch the two-dimensional phase plane and one-dimensional phase line for the decoupled systems, and
(d) give a rough sketch of the phase portrait of the system.

4. $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

5. $\mathbf{A} = \begin{pmatrix} -2 & 3 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

6. $\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$

7. $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

Exercises 8–9 consider the properties of the cubic polynomial

$$p(\lambda) = \alpha\lambda^3 + \beta\lambda^2 + \gamma\lambda + \delta,$$

where α, β, γ, and δ are real numbers.

8. **(a)** Show that, if α is positive, then the limit of $p(\lambda)$ as $\lambda \to \infty$ is ∞ and the limit of $p(\lambda)$ as $\lambda \to -\infty$ is $-\infty$.

(b) Show that, if α is negative, then the limit of $p(\lambda)$ as $\lambda \to \infty$ is $-\infty$ and the limit of $p(\lambda)$ as $\lambda \to -\infty$ is ∞.

(c) Using the above, show that $p(\lambda)$ must have at least one real root (that is, at least one real number λ_0 such that $p(\lambda_0) = 0$). [*Hint*: Look at the graph of $p(\lambda)$.]

9. Suppose $a + ib$ is a root of $p(\lambda)$ (so $p(a + ib) = 0$). Show that $a - ib$ is also a root. [*Hint*: Remember that a complex number is zero if and only if both its real and imaginary parts are zero. Then compute $p(a + ib)$ and $p(a - ib)$.]

In Exercises 10–13, consider the linear system $d\mathbf{Y}/dt = \mathbf{BY}$ with the coefficient matrix $\mathbf{B}$ specified. These systems do not fit into the classification of the most common types of systems given in the text. However, the equations for dx/dt and dy/dt decouple from dz/dt. For each of these systems,

(a) compute the eigenvalues,
(b) sketch the xy-phase plane and the z-phase line, and
(c) give a rough sketch of the phase portrait of the system.

10. $\mathbf{B} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

11. $\mathbf{B} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

12. $\mathbf{B} = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

13. $\mathbf{B} = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

In Exercises 14–15, consider the linear system $d\mathbf{Y}/dt = \mathbf{C}\,\mathbf{Y}$. These systems do not fit into the classification of the most common types of systems given in the text, and they do not decouple into lower-dimensional systems. For each system,

(a) compute the eigenvalues,
(b) compute the eigenvectors, and
(c) sketch (as best you can) the phase portrait of the system. [*Hint*: Use the eigenvalues and eigenvectors and also vectors in the vector field.]

14. $\mathbf{C} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}$

15. $\mathbf{C} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

16. For the linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY} = \begin{pmatrix} 2 & -1 & 0 \\ 0 & -2 & 3 \\ -1 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}:$$

(a) Show that $\mathbf{V}_1 = (1, 1, 1)$ is an eigenvector of the coefficient matrix by computing $\mathbf{AV}_1$. What is the eigenvalue for this eigenvector?

(b) Find the other two eigenvalues for the matrix $\mathbf{A}$.

(c) Classify the system (source, sink, ...).

(d) Sketch (as best you can) the phase portrait. [*Hint*: Use the other eigenvalues and find the other eigenvectors.]

17. For the linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY} = \begin{pmatrix} -4 & 3 & 0 \\ 0 & -1 & 1 \\ 5 & -5 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}:$$

(a) Show that $\mathbf{V}_1 = (1, 1, 0)$ is an eigenvector of the coefficient matrix by computing $\mathbf{AV}_1$. What is the eigenvalue for this eigenvector?

(b) Find the other two eigenvalues for the matrix $\mathbf{A}$.

(c) Classify the system (source, sink, ...).

(d) Sketch (as best you can) the phase portrait. [*Hint*: Use the other eigenvalues and find the other eigenvectors.]

18. Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{BY} = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(This system is related to the Lorenz system studied in Section 2.8, and we will use the results obtained in this exercise when we return to the Lorenz equations in Section 5.5.)

(a) Find the characteristic polynomial and the eigenvalues.

(b) Find the eigenvectors.

(c) Sketch the phase portrait (as best you can).

(d) Comment on how the fact that the system "decouples" helps in the computations and in sketching the phase space.

Many years later, when Glen finally retires from writing math texts, he decides to join his friends and former collaborators Paul and Bob. He opens an ice creme store between Paul's and Bob's cafés. Let $z(t)$ be Glen's profits at time t (with $x(t)$ and $y(t)$ representing Paul's and Bob's profits, respectively). Suppose the three stores affect each other in such a way that

$$\begin{aligned} \frac{dx}{dt} &= -y + z \\ \frac{dy}{dt} &= -x + z \\ \frac{dz}{dt} &= z. \end{aligned}$$

19. **(a)** If Glen makes a profit, does this help or hurt Paul's and Bob's profits?

(b) If Paul and Bob are making profits, does this help or hurt Glen's profits?

20. Write this system in matrix form and find the eigenvalues. Use them to classify the system.

21. Suppose that at time $t = 0$, both Paul and Bob are making (equal) small profits, but Glen is just breaking even [$x(0) = y(0)$ are small and positive, but $z(0) = 0$].

(a) Sketch the solution curve in the xyz-phase space.

(b) Sketch the $x(t)$-, $y(t)$-, and $z(t)$-graphs of the solution.

(c) Describe what happens to the profits of each store.

22. Suppose that at time $t = 0$ both Paul and Bob are just breaking even, but Glen is making a small profit [$x(0)$ and $y(0)$ are zero, but $z(0)$ is small and positive].

(a) Sketch the solution curve in the xyz-phase space.

(b) Sketch the $x(t)$-, $y(t)$-, and $z(t)$-graphs of the solution.

(c) Describe what happens to the profits of each store.