EXERCISES FOR SECTION 3.7

1. Construct a table of the possible linear systems as follows:

(a) The first column contains the type of the system (sink, spiral sink, source, ...), if it has a name.

(b) The second column contains the condition on the eigenvalues that corresponds to this case.

(c) The third column contains a small picture of two or more possible phase portraits for this system, and

(d) The fourth column contains $x(t)$- and $y(t)$-graphs of typical solutions indicated in your phase portraits.

[*Hint*: The most complete table contains 14 cases. Don't forget the double eigenvalue and zero eigenvalue cases.]

In Exercises 2–7, we consider the one-parameter families of linear systems depending on the parameter a. Each family therefore determines a curve in the trace-determinant plane. For each family,

(a) sketch the corresponding curve in the trace-determinant plane;

(b) in a brief essay, discuss different types of behaviors exhibited by the system as a increases along the real line (unless otherwise noted); and

(c) identify the values of a where the type of the system changes. These are the bifurcation values of a.

2. $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & -1 \\ 2 & 0 \end{pmatrix} \mathbf{Y}$

3. $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & a^2 + a \\ 1 & a \end{pmatrix} \mathbf{Y}$

4. $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & a \\ 1 & 0 \end{pmatrix} \mathbf{Y}$

5. $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & \sqrt{1 - a^2} \\ 1 & 0 \end{pmatrix} \mathbf{Y}$

$-1 \leq a \leq 1$

6. $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 0 \\ a & -3 \end{pmatrix} \mathbf{Y}$

7. $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & 1 \\ a & a \end{pmatrix} \mathbf{Y}$

8. Consider the two-parameter family of linear systems

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & 1 \\ b & 1 \end{pmatrix} \mathbf{Y}.$$

In the ab-plane, identify all regions where this system possesses a saddle, a sink, a spiral sink, and so on. [*Hint*: Draw a picture of the ab-plane and shade each point (a, b) of the plane a different color depending on the type of linear system for that choice (a, b) of parameters.]

9. Consider the two-parameter family of linear systems

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mathbf{Y}.$$

In the ab-plane, identify all regions where this system possesses a saddle, a sink, a spiral sink, and so on. [*Hint*: Draw a picture of the ab-plane and shade each point (a, b) of the plane a different color depending on the type of linear system for that choice (a, b) of the parameters.]

10. Consider the two-parameter family of linear systems

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mathbf{Y}.$$

In the ab-plane, identify all regions where this system possesses a saddle, a sink, a spiral sink, and so on. [*Hint*: Draw a picture of the ab-plane and shade each point (a, b) of the plane a different color depending on the type of linear system for that choice (a, b) of parameters.]

In Exercises 11–13, we consider the equation

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

that models the motion of a harmonic oscillator with mass m, spring constant k, and damping coefficient b. In each exercise, we fix two values of these three parameters and obtain a one-parameter family of second-order equations. For each one-parameter family,

(a) rewrite the one-parameter family as a one-parameter family of linear systems,

(b) draw the curve in the trace-determinant plane obtained by varying the parameter, and

(c) in a brief essay, discuss the different types of behavior exhibited by this one-parameter family.

11. Consider

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + 3y = 0.$$

That is, fix $m = 1$ and $k = 3$, and let $0 \le b < \infty$.

12. Consider

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + ky = 0.$$

That is, fix $m = 1$ and $b = 2$, and let $0 < k < \infty$.

13. Consider

$$m\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2y = 0.$$

That is, fix $b = 1$ and $k = 2$, and let $0 < m < \infty$.

14. Using the `DETools` program `TDPlaneQuiz`, describe the path through the trace-determinant plane that was used to produce each animation.

3.8 LINEAR SYSTEMS IN THREE DIMENSIONS

So far, we have studied linear systems with two dependent variables. For these systems, the behavior of solutions and the nature of the phase plane can be determined by computing the eigenvalues and eigenvectors of the 2×2 coefficient matrix. Once we have found two solutions with linearly independent initial conditions, we can give the general solution.

In this section we show that the same is true for linear systems with three dependent variables. The eigenvalues and eigenvectors of the 3×3 coefficient matrix determine the behavior of solutions and the general solution. Three-dimensional linear systems have three eigenvalues, so the list of possible qualitatively distinct phase spaces is longer than for planar systems. Since we must deal with three scalar equations rather than two, the arithmetic can quickly become much more involved. You might want to seek out software or a calculator capable of handling 3×3 matrices.

Linear Independence and the Linearity Principle

The general form of a linear system with three dependent variables is

$$\begin{aligned}\frac{dx}{dt} &= a_{11}x + a_{12}y + a_{13}z\\ \frac{dy}{dt} &= a_{21}x + a_{22}y + a_{23}z\\ \frac{dz}{dt} &= a_{31}x + a_{32}y + a_{33}z,\end{aligned}$$

where x, y, and z are the dependent variables and the coefficients a_{ij}, $(i, j = 1, 2, 3)$, are constants. We can write this system in matrix form as

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY},$$

where $\mathbf{A}$ is the coefficient matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$$