EXERCISES FOR SECTION 3.5

In Exercises 1–4, each of the linear systems has one eigenvalue and one line of eigenvectors. For each system,

(a) find the eigenvalue;

(b) find an eigenvector;

(c) sketch the direction field;

(d) sketch the phase portrait, including the solution curve with initial condition $\mathbf{Y}_0 = (1, 0)$; and

(e) sketch the $x(t)$- and $y(t)$-graphs of the solution with initial condition $\mathbf{Y}_0 = (1, 0)$.

1. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 0 \\ 1 & -3 \end{pmatrix} \mathbf{Y}$

2. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \mathbf{Y}$

3. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} \mathbf{Y}$

4. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \mathbf{Y}$

In Exercises 5–8, the linear systems are the same as those in Exercises 1–4. For each system,

(a) find the general solution;

(b) find the particular solution for the initial condition $\mathbf{Y}_0 = (1, 0)$; and

(c) sketch the $x(t)$- and $y(t)$-graphs of the solution. (Compare these sketches with the sketches you obtained in the corresponding problem from Exercises 1–4.)

5. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 0 \\ 1 & -3 \end{pmatrix} \mathbf{Y}$

6. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \mathbf{Y}$

7. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} \mathbf{Y}$

8. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \mathbf{Y}$

9. Given a quadratic $\lambda^2 + \alpha\lambda + \beta$, what condition on α and β guarantees

(a) that the quadratic has a double root?

(b) that the quadratic has zero as a root?

10. Evaluate the limit of $te^{\lambda t}$ as $t \to \infty$ if

(a) $\lambda > 0$

(b) $\lambda < 0$

Be sure to justify your answer.

11. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix},$$

where p and q are positive. What condition on q and p guarantees:

(a) that A has two real eigenvalues?

(b) that A has complex eigenvalues?

(c) that A has only one eigenvalue and one line of eigenvectors?

12. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Define the trace of $\mathbf{A}$ to be $\operatorname{tr}(\mathbf{A}) = a + d$. Show that $\mathbf{A}$ has only one eigenvalue if and only if $(\operatorname{tr}(\mathbf{A}))^2 - 4\det(\mathbf{A}) = 0$.

13. Suppose

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a matrix with eigenvalue λ such that every nonzero vector is an eigenvector with eigenvalue λ, that is, $\mathbf{AY} = \lambda\mathbf{Y}$ for every vector $\mathbf{Y}$. Show that $a = d = \lambda$ and $b = c = 0$. [*Hint*: Since $\mathbf{AY} = \lambda\mathbf{Y}$ for every $\mathbf{Y}$, try $\mathbf{Y} = (1, 0)$ and $\mathbf{Y} = (0, 1)$.]

14. Suppose λ is an eigenvalue for the matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and suppose that there are two linearly independent eigenvectors $\mathbf{Y}_1$ and $\mathbf{Y}_2$ associated to λ. Show that every nonzero vector is an eigenvector with eigenvalue λ. What does this imply about a, b, c, and d?

15. Suppose the two functions

$$\mathbf{Y}_1(t) = e^{\lambda t}\mathbf{V}_0 + te^{\lambda t}\mathbf{V}_1 \quad \text{and} \quad \mathbf{Y}_2(t) = e^{\lambda t}\mathbf{W}_0 + te^{\lambda t}\mathbf{W}_1$$

are equal for all t. Show that $\mathbf{V}_0 = \mathbf{W}_0$ and $\mathbf{V}_1 = \mathbf{W}_1$.

16. Suppose λ_0 is a repeated eigenvalue for the 2×2 matrix $\mathbf{A}$.

(a) Show that $(\mathbf{A} - \lambda_0\mathbf{I})^2 = \mathbf{0}$ (the zero matrix).

(b) Given an arbitrary vector $\mathbf{V}_0$, let $\mathbf{V}_1 = (\mathbf{A} - \lambda_0\mathbf{I})\mathbf{V}_0$. Using the result of part (a), show that $\mathbf{V}_1$ is either an eigenvector of $\mathbf{A}$ or the zero vector.

In Exercises 17–19, each of the given linear systems has zero as an eigenvalue. For each system,

(a) find the eigenvalues;

(b) find the eigenvectors;

(c) sketch the phase portrait;

(d) sketch the $x(t)$- and $y(t)$-graphs of the solution with initial condition $\mathbf{Y}_0 = (1, 0)$;

(e) find the general solution; and

(f) find the particular solution for the initial condition $\mathbf{Y}_0 = (1, 0)$ and compare it with your sketch from part (d).

17. $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix} \mathbf{Y}$ **18.** $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} \mathbf{Y}$ **19.** $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{Y}$

20. Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) Show that if one or both of the eigenvalues of $\mathbf{A}$ is zero, then the determinant of $\mathbf{A}$ is zero.

(b) Show that if $\det \mathbf{A} = 0$, then at least one of the eigenvalues of $\mathbf{A}$ is zero.

21. Find the eigenvalues and sketch the phase portraits for the linear systems

(a) $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \mathbf{Y}$ **(b)** $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \mathbf{Y}$

22. Find the general solution for the linear systems

(a) $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \mathbf{Y}$ **(b)** $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \mathbf{Y}$

23. Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mathbf{Y}.$$

(a) Find the eigenvalues.

(b) Find the eigenvectors.

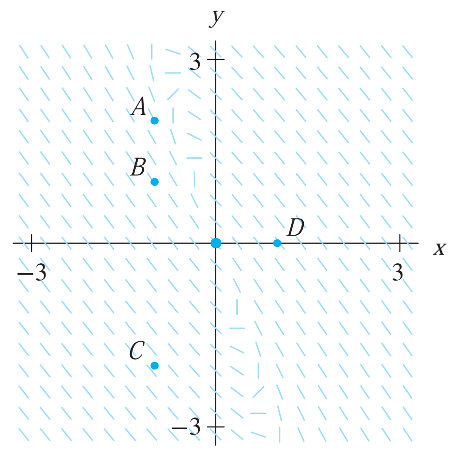
(c) Suppose $a = d < 0$. Sketch the phase portrait and compute the general solution. (What are the eigenvectors in this case?)

(d) Suppose $a = d > 0$. Sketch the phase portrait and compute the general solution.

24. The slope field for the system

$$\frac{dx}{dt} = -3x - y$$
$$\frac{dy}{dt} = 4x + y$$

is shown at the right.



(a) Determine the type of the equilibrium point at the origin.

(b) Calculate all straight-line solutions.

(c) Plot the $x(t)$- and $y(t)$-graphs ($t \geq 0$) for the initial conditions $A = (-1, 2)$, $B = (-1, 1)$, $C = (-1, -2)$, and $D = (1, 0)$.

3.6 SECOND-ORDER LINEAR EQUATIONS

Throughout this chapter we have used the harmonic oscillator as an example. We have solved the second-order equation and its associated system of equations in a number of different cases. Now it is time to summarize all that we have learned about this important model.

Second-Order Equations versus First-Order Systems

As we know, the motion of a harmonic oscillator can be modeled by the second-order equation

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0,$$

where $m > 0$ is the mass, $k > 0$ is the spring constant, and $b \geq 0$ is the damping coefficient. Since $m \neq 0$, we can also write this equation in the form

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0,$$

where $p = b/m$ and $q = k/m$ are nonnegative constants, and the corresponding linear system is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \mathbf{Y}.$$

As we will see in this section, any method to compute the general solution of the second-order equation also gives the general solution of the associated system, and vice versa. In particular we can use the Linearity Principle to produce new solutions from