determine the direction (clockwise or counterclockwise) and approximate shape of the solution curves by sketching the phase portrait (see Figure 3.29).

The $x(t)$- and $y(t)$-graphs of solutions oscillate with increasing amplitude. The period of these oscillations is $2\pi/(\sqrt{7}/2) = 4\pi/\sqrt{7} \approx 4.71$, and the amplitude increases like $e^{t/2}$. We sketch the qualitative behavior of the $x(t)$- and $y(t)$-graphs in Figure 3.30.

Either Paul and Bob will stay precisely at the break-even point $(x, y) = (0, 0)$, or the profits and losses of their cafés will go up and down with increasing amplitude (a boom to bust to boom business cycle). Also the equilibrium point at the origin is unstable, so even a tiny profit or loss by either café eventually leads to large oscillations in the profits of both cafés. It would be very difficult to predict this behavior from just looking at the linear system without any computations.

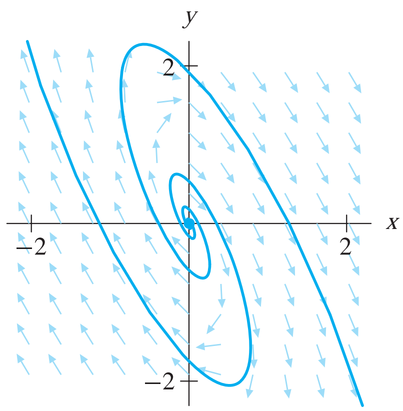


Figure 3.29
Phase portrait for

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 1 \\ -4 & -1 \end{pmatrix} \mathbf{Y}.$$

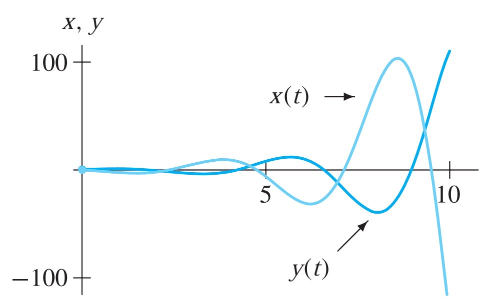


Figure 3.30
$x(t)$- and $y(t)$- graphs of a solution for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 1 \\ -4 & -1 \end{pmatrix} \mathbf{Y}.$$

EXERCISES FOR SECTION 3.4

1. Suppose that the 2×2 matrix $\mathbf{A}$ has $\lambda = 1 + 3i$ as an eigenvalue with eigenvector

$$\mathbf{Y}_0 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}.$$

Compute the general solution to $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$.

2. Suppose that the 2×2 matrix $\mathbf{B}$ has $\lambda = -2 + 5i$ as an eigenvalue with eigenvector

$$\mathbf{Y}_0 = \begin{pmatrix} 1 \\ 4-3i \end{pmatrix}.$$

Compute the general solution to $d\mathbf{Y}/dt = \mathbf{B}\mathbf{Y}$.

In Exercises 3–8, each linear system has complex eigenvalues. For each system,

(a) find the eigenvalues;

(b) determine if the origin is a spiral sink, a spiral source, or a center;

(c) determine the natural period and natural frequency of the oscillations,

(d) determine the direction of the oscillations in the phase plane (do the solutions go clockwise or counterclockwise around the origin?); and

(e) using HPGSystemSolver, sketch the xy-phase portrait and the $x(t)$- and $y(t)$-graphs for the solutions with the indicated initial conditions.

3. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \mathbf{Y}$, with initial condition $\mathbf{Y}_0 = (1, 0)$

4. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix} \mathbf{Y}$, with initial condition $\mathbf{Y}_0 = (1, 1)$.

5. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & -5 \\ 3 & 1 \end{pmatrix} \mathbf{Y}$, with initial condition $\mathbf{Y}_0 = (4, 0)$

6. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 2 \\ -2 & -1 \end{pmatrix} \mathbf{Y}$, with initial condition $\mathbf{Y}_0 = (-1, 1)$

7. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & -6 \\ 2 & 1 \end{pmatrix} \mathbf{Y}$, with initial condition $\mathbf{Y}_0 = (2, 1)$

8. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 4 \\ -3 & 2 \end{pmatrix} \mathbf{Y}$, with initial condition $\mathbf{Y}_0 = (1, -1)$

In Exercises 9–14, the linear systems are the same as in Exercises 3–8. For each system,

(a) find the general solution;

(b) find the particular solution with the given initial value; and

(c) sketch the $x(t)$- and $y(t)$-graphs of the particular solution. (Compare these sketches with the sketches you obtained in the corresponding problem from Exercises 3–8.)

9. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \mathbf{Y}$, with initial condition $\mathbf{Y}_0 = (1, 0)$

10. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix} \mathbf{Y}$, with initial condition $\mathbf{Y}_0 = (1, 1)$.

11. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & -5 \\ 3 & 1 \end{pmatrix} \mathbf{Y}$, with initial condition $\mathbf{Y}_0 = (4, 0)$

12. $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 2 \\ -2 & -1 \end{pmatrix} \mathbf{Y}$, with initial condition $\mathbf{Y}_0 = (-1, 1)$

13. $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & -6 \\ 2 & 1 \end{pmatrix} \mathbf{Y}$, with initial condition $\mathbf{Y}_0 = (2, 1)$

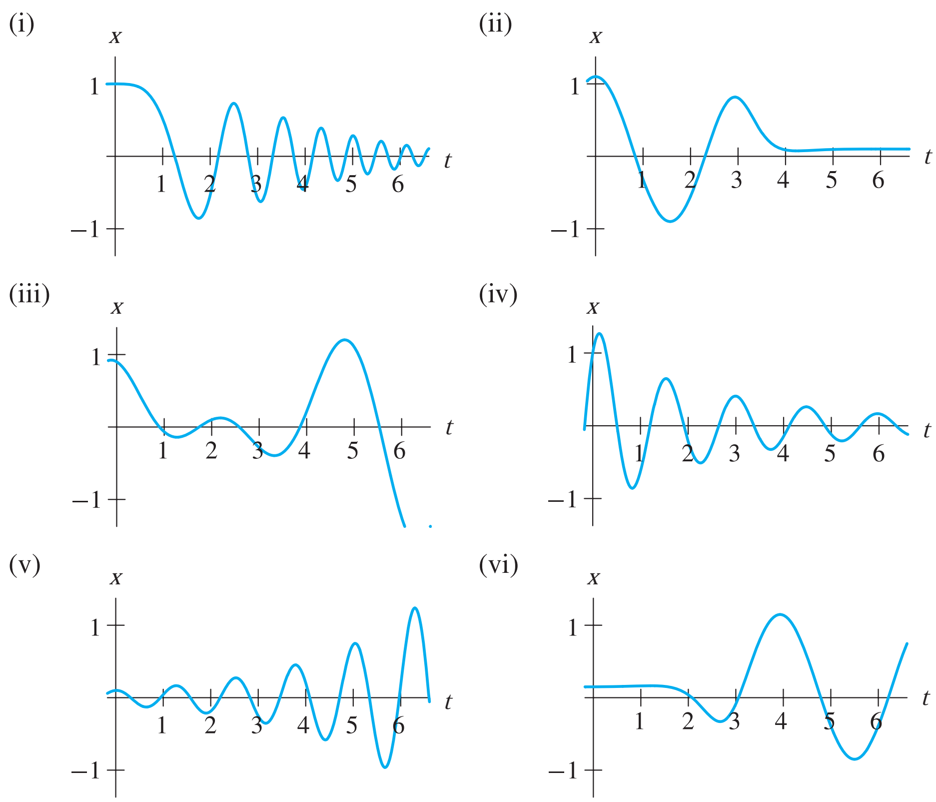
14. $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 4 \\ -3 & 2 \end{pmatrix} \mathbf{Y}$, with initial condition $\mathbf{Y}_0 = (1, -1)$

15. The following six figures are graphs of functions $x(t)$.

(a) Which of the graphs can be $x(t)$-graphs for a solution of a linear system with complex eigenvalues?

(b) For each such graph, give the natural period of the system and classify the equilibrium point at the origin as a spiral sink, a spiral source, or a center.

(c) For each graph that cannot be an $x(t)$-graph for a solution of a linear system with complex eigenvalues, explain why not.



16. Show that a matrix of the form

$$\mathbf{A} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

with $b \neq 0$ has complex eigenvalues.

17. Suppose that a and b are real numbers and that the polynomial $\lambda^2 + a\lambda + b$ has $\lambda_1 = \alpha + i\beta$ as a root with $\beta \neq 0$. Show that $\lambda_2 = \alpha - i\beta$, the complex conjugate of λ_1, must also be a root. [*Hint*: There are (at least) two ways to attack this problem. Either look at the form of the quadratic formula for the roots, or notice that

$$(\alpha + i\beta)^2 + a(\alpha + i\beta) + b = 0$$

and take the complex conjugate of both sides of this equation.]

18. Suppose that the matrix $\mathbf{A}$ with real entries has complex eigenvalues $\lambda = \alpha + i\beta$ and $\overline{\lambda} = \alpha - i\beta$ with $\beta \neq 0$. Show that the eigenvectors of $\mathbf{A}$ must be complex; that is, show that, if $\mathbf{Y}_0 = (x_0, y_0)$ is an eigenvector for $\mathbf{A}$, then either x_0 or y_0 or both have a nonzero imaginary part.

19. Suppose the matrix $\mathbf{A}$ with real entries has the complex eigenvalue $\lambda = \alpha + i\beta$, $\beta \neq 0$. Let $\mathbf{Y}_0$ be an eigenvector for λ and write $\mathbf{Y}_0 = \mathbf{Y}_1 + i\mathbf{Y}_2$, where $\mathbf{Y}_1 = (x_1, y_1)$ and $\mathbf{Y}_2 = (x_2, y_2)$ have real entries. Show that $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are linearly independent. [*Hint*: Suppose they are not linearly independent. Then $(x_2, y_2) = k(x_1, y_1)$ for some constant k. Then $\mathbf{Y}_0 = (1 + ik)\mathbf{Y}_1$. Then use the fact that $\mathbf{Y}_0$ is an eigenvector of $\mathbf{A}$ and that $\mathbf{A}\mathbf{Y}_1$ contains no imaginary part.]

20. Suppose the matrix $\mathbf{A}$ with real entries has complex eigenvalues $\lambda = \alpha + i\beta$ and $\overline{\lambda} = \alpha - i\beta$. Suppose also that $\mathbf{Y}_0 = (x_1 + iy_1, x_2 + iy_2)$ is an eigenvector for the eigenvalue λ. Show that $\overline{\mathbf{Y}_0} = (x_1 - iy_1, x_2 - iy_2)$ is an eigenvector for the eigenvalue $\overline{\lambda}$. In other words, the complex conjugate of an eigenvector for λ is an eigenvector for $\overline{\lambda}$.

21. Consider the function $x(t) = e^{-\alpha t} \sin \beta t$, where α and β are positive.

(a) What is the distance between successive zeros of this function? More precisely, if $t_1 < t_2$ are such that $x(t_1) = x(t_2) = 0$ and $x(t) \neq 0$ for $t_1 < t < t_2$, then what is $t_2 - t_1$?

(b) What is the distance between the first local maximum and the first local minimum of $x(t)$ for $t > 0$?

(c) What is the distance between the first two local maxima of $x(t)$ for $t > 0$?

(d) What is the distance between $t = 0$ and the first local maximum of $x(t)$ for $t > 0$?

22. Show that a function of the form

$$x(t) = k_1 \cos \beta t + k_2 \sin \beta t$$

can be written as

$$x(t) = K \cos(\beta t - \phi),$$

where $K = \sqrt{k_1^2 + k_2^2}$. (Sometimes a solution of a linear system with complex coefficients is expressed in this form in order to clarify its behavior. The magnitude K gives the *amplitude* of the solution, and the angle ϕ is the *phase* of the solution.) [*Hint*: Pick ϕ such that $K \cos\phi = k_1$ and $K \sin\phi = k_2$.]

23. For the second-order equation

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0 :$$

(a) Write this equation as a first-order linear system.

(b) What conditions on p and q guarantee that the eigenvalues of the corresponding linear system are complex?

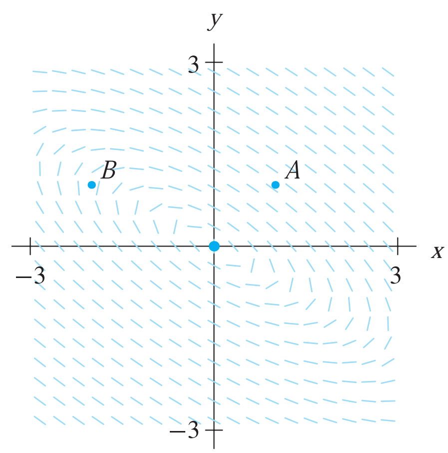
(c) What relationship between p and q guarantees that the origin is a spiral sink? What relationship guarantees that the origin is a center? What relationship guarantees that the origin is a spiral source?

(d) If the eigenvalues are complex, what conditions on p and q guarantee that solutions spiral around the origin in a clockwise direction?

24. The slope field for the system

$$\frac{dx}{dt} = -0.9x - 2y$$
$$\frac{dy}{dt} = x + 1.1y$$

is given to the right. Plot the $x(t)$- and $y(t)$-graphs for the initial conditions $A = (1, 1)$ and $B = (-2, 1)$. What do the graphs have in common?



25. (Essay Question) We have seen that linear systems with real eigenvalues can be classified as sinks, sources, or saddles, depending on whether the eigenvalues are greater or less than zero. Linear systems with complex eigenvalues can be classified as spiral sources, spiral sinks, or centers, depending on the sign of the real part of the eigenvalue. Why is there not a type of linear system called a "spiral saddle"?

26. Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 10 \\ -1 & 3 \end{pmatrix}\mathbf{Y}.$$

Show that all solution curves in the phase portrait for this system are elliptical.