That is, $\mathbf{Y}_1(0)$ lies on the y-axis and $\mathbf{Y}_2(t)$ lies on the v-axis. Consequently, these vectors are linearly independent, and the general solution to the first-order system is

$$\mathbf{Y}(t) = k_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + k_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

Recalling that solutions $\mathbf{Y}(t)$ to the first-order system are really functions of the form $\mathbf{Y}(t) = (y(t), v(t))$ where $y(t)$ is a solution to the original second-order equation, we obtain the general solution to the original second-order equation using the first component of $\mathbf{Y}(t)$. The result is

$$y(t) = k_1 \cos t + k_2 \sin t.$$

In Section 3.6 we will discuss a more immediate way to find the general solution of second-order equations such as this one, but it is important to realize that the Linearity Principle also applies to "linear" second-order equations such as the equation for a damped harmonic oscillator.

EXERCISES FOR SECTION 3.1

Recall the model

$$\begin{aligned}\frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy,\end{aligned}$$

for Paul's and Bob's cafés, where $x(t)$ is Paul's daily profit, $y(t)$ is Bob's daily profit, and a, b, c, and d are parameters governing how the daily profit of each store affects the other. In Exercises 1–4, different choices of the parameters a, b, c, and d are specified. For each exercise write a brief paragraph describing the interaction between the stores, given the specified parameter values. [For example, suppose $a = 1$, $c = -1$, and $b = d = 0$. If Paul's store is making a profit ($x > 0$), then Paul's profit increases more quickly (because $ax > 0$). However, if Paul makes a profit, then Bob's profits suffer (because $cx < 0$). Since $b = d = 0$, Bob's current profits have no impact on his or Paul's future profits.]

1. $a = 1$, $b = -1$, $c = 1$, and $d = -1$

2. $a = 2$, $b = -1$, $c = 0$, and $d = 0$

3. $a = 1$, $b = 0$, $c = 2$, and $d = 1$

4. $a = -1$, $b = 2$, $c = 2$, and $d = -1$

In Exercises 5–7, rewrite the specified linear system in matrix form.

5. $$\begin{aligned}\frac{dx}{dt} &= 2x + y \\ \frac{dy}{dt} &= x + y\end{aligned}$$

6. $$\begin{aligned}\frac{dx}{dt} &= 3y \\ \frac{dy}{dt} &= 3\pi y - 0.3x\end{aligned}$$

7. $$\begin{aligned}\frac{dp}{dt} &= 3p - 2q - 7r \\ \frac{dq}{dt} &= -2p + 6r \\ \frac{dr}{dt} &= 7.3q + 2r\end{aligned}$$

In Exercises 8–9, rewrite the specified linear system in component form.

8. $$\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -3 & 2\pi \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

9. $$\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ \gamma & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

For the linear systems given in Exercises 10–13, use HPGSystemSolver to sketch the direction field, several solutions, and the $x(t)$- and $y(t)$-graphs for the solution with initial condition $(x, y) = (1, 1)$.

10. $$\frac{dx}{dt} = 2x + y$$ $$\frac{dy}{dt} = x + y$$

11. $$\frac{dx}{dt} = x + 2y$$ $$\frac{dy}{dt} = -x - y$$

12. $$\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -3 & 2\pi \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

13. $$\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -1 & -11 \\ 6 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

14. Let
$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
be a nonsingular matrix ($\det \mathbf{A} \neq 0$).

(a) Show that, if $a = 0$, then $b \neq 0$ and $c \neq 0$.

(b) Suppose $a = 0$. Use the result of part (a) to show that the origin is the only equilibrium point.

Along with the verification given in the section, this result shows that, if $\det \mathbf{A} \neq 0$, then the only equilibrium point for the system $d\mathbf{Y}/dt = \mathbf{AY}$ is the origin.

15. Let
$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
be a nonzero matrix. That is, suppose that at least one of its entries is nonzero. Show that, if $\det \mathbf{A} = 0$, then the system $d\mathbf{Y}/dt = \mathbf{AY}$ has an entire line of equilibria. [*Hint*: First consider the case where $a \neq 0$. Show that any point (x_0, y_0) that satisfies $x_0 = (-b/a)y_0$ is an equilibrium point. What if we assume that entries of $\mathbf{A}$ other than a are nonzero?]

16. The general form of a linear, homogeneous, second-order equation with constant coefficients is
$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0.$$

(a) Write the first-order system for this equation, and write this system in matrix form.

(b) Show that if $q \neq 0$, then the origin is the only equilibrium point of the system.

(c) Show that if $q \neq 0$, then the only solution of the second-order equation with y constant is $y(t) = 0$ for all t.

17. Consider the linear system corresponding to the second-order equation

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0.$$

(a) If $q = 0$ and $p \neq 0$, find all the equilibrium points.

(b) If $q = p = 0$, find all the equilibrium points.

18. Convert the second-order equation

$$\frac{d^2y}{dt^2} = 0$$

into a first-order system using $v = dy/dt$ as usual.

(a) Find the general solution for the dv/dt equation.

(b) Substitute this solution into the dy/dt equation, and find the general solution of the system.

(c) Sketch the phase portrait of the system.

19. Convert the third-order differential equation

$$\frac{d^3y}{dt^3} + p\frac{d^2y}{dt^2} + q\frac{dy}{dt} + ry = 0,$$

where p, q, and r are constants, to a three-dimensional linear system written in matrix form.

In Exercises 20–23, we consider the following model of the market for single-family housing in a community. Let $S(t)$ be the number of sellers at time t, and let $B(t)$ be the number of buyers at time t. We assume that there are natural equilibrium levels of buyers and sellers (made up of people who retire, change job locations, or wish to move for family reasons). The equilibrium level of sellers is S_0, and the equilibrium level of buyers is B_0.

However, market forces can entice people to buy or sell under various conditions. For example, if the price of a house is very high, then house owners are tempted to sell their homes. If prices are very low, extra buyers enter the market looking for bargains. We let $b(t) = B(t) - B_0$ denote the deviation of the number of buyers from equilibrium at time t. So if $b(t) > 0$, then there are more buyers than usual, and we say it is a "seller's market." Presumably the competition of the extra buyers for the same number of houses for sale will force the prices up (the law of supply and demand).

Similarly, we let $s(t) = S(t) - S_0$ denote the deviation of the number of sellers from the equilibrium level. If $s(t) > 0$, then there are more sellers on the market than usual; and if the number of buyers is low, there are too many houses on the market and prices decrease, which in turn affects decisions to buy or sell.

We can give a simple model of this situation as follows:

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} b \\ s \end{pmatrix}, \quad \text{where } \mathbf{Y} = \begin{pmatrix} b \\ s \end{pmatrix}.$$

The exact values of the parameters α, β, γ, and δ depend on the economy of a particular community. Nevertheless, if we assume that everybody wants to get a bargain when they are buying a house and to get top dollar when they are selling a house, then we can hope to predict whether the parameters are positive or negative even though we cannot predict their exact values.

Use the information given above to obtain information about the parameters α, β, γ, and δ. Be sure to justify your answers.

20. If there are more than the usual number of buyers competing for houses, we would expect the price of houses to rise, and this increase would make it less likely that new potential buyers will enter the market. What does this say about the parameter α?

21. If there are fewer than the usual number of buyers competing for the houses available for sale, then we would expect the price of houses to decrease. As a result, fewer potential sellers will place their houses on the market. What does this imply about the parameter γ?

22. Consider the effect on house prices if $s > 0$ and the subsequent effect on buyers and sellers. Then determine the sign of the parameter β.

23. Determine the most reasonable sign for the parameter δ.

24. Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{Y}.$$

(a) Show that the two functions

$$\mathbf{Y}_1(t) = \begin{pmatrix} 0 \\ e^t \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

are solutions to the differential equation.

(b) Solve the initial-value problem

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

25. Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{Y}.$$

(a) Show that the function

$$\mathbf{Y}(t) = \begin{pmatrix} te^{2t} \\ -(t+1)e^{2t} \end{pmatrix}$$

is a solution to the differential equation.

(b) Solve the initial-value problem

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

In Exercises 26–29, a coefficient matrix for the linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}, \quad \text{where } \mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

is specified. Also two functions and an initial value are given. For each system:

(a) Check that the two functions are solutions of the system; if they are not solutions, then stop.

(b) Check that the two solutions are linearly independent; if they are not linearly independent, then stop.

(c) Find the solution to the linear system with the given initial value.

26. $\mathbf{A} = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix}$

Functions: $\mathbf{Y}_1(t) = (e^{-3t}, e^{-3t})$
$\mathbf{Y}_2(t) = (e^{-4t}, 2e^{-4t})$
Initial value: $\mathbf{Y}(0) = (2, 3)$

27. $\mathbf{A} = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix}$

Functions: $\mathbf{Y}_1(t) = (e^{-3t} - 2e^{-4t}, e^{-3t} - 4e^{-4t})$
$\mathbf{Y}_2(t) = (2e^{-3t} + e^{-4t}, 2e^{-3t} + 2e^{-4t})$
Initial value: $\mathbf{Y}(0) = (2, 3)$

28. $\mathbf{A} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}$

Functions: $\mathbf{Y}_1(t) = e^{-2t}(\cos 3t, \sin 3t)$
$\mathbf{Y}_2(t) = e^{-2t}(-\sin 3t, \cos 3t)$
Initial value: $\mathbf{Y}(0) = (2, 3)$

29. $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$

Functions: $\mathbf{Y}_1(t) = (-e^{-t} + 12e^{3t}, e^{-t} + 4e^{3t})$
$\mathbf{Y}_2(t) = (-e^{-t}, 2e^{-t})$
Initial value: $\mathbf{Y}(0) = (2, 3)$

30. **(a)** Verify property 1, $\mathbf{A}\, k\mathbf{Y} = k\mathbf{A}\mathbf{Y}$, of matrix multiplication, where $\mathbf{Y}$ is a (two-dimensional) vector, $\mathbf{A}$ is a matrix, and k is a constant.

(b) Using scalar notation, write out and verify the Linearity Principle. (Aren't matrices nice?)

31. Show that the vectors (x_1, y_1) and (x_2, y_2) are linearly dependent—that is, not linearly independent—if any of the following conditions are satisfied.

(a) If $(x_1, y_1) = (0, 0)$.

(b) If $(x_1, y_1) = \lambda(x_2, y_2)$ for some constant λ.

(c) If $x_1 y_2 - x_2 y_1 = 0$. *Hint*: Assume x_1 is not zero; then $y_2 = x_2 y_1 / x_1$. But $x_2 = x_2 x_1 / x_1$, and we can use part b. The other cases are similar. Note that the quantity $x_1 y_2 - x_2 y_1$ is the determinant of the matrix

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}.$$

32. Given the vectors (x_1, y_1) and (x_2, y_2), show that they are linearly independent if the quantity $x_1 y_2 - x_2 y_1$ is nonzero (see part (c) of Exercise 31). [*Hint*: Suppose $x_2 \neq 0$. If (x_1, y_1) and (x_2, y_2) are on the same line through $(0, 0)$, then $(x_1, y_1) = \lambda(x_2, y_2)$ for some λ. But then $\lambda = x_1/x_2$ and $\lambda = y_1/y_2$. What does this say about x_1/x_2 and y_1/y_2? What if $x_2 = 0$?]

33. Suppose that $\mathbf{Y}_1(t) = (-e^{-t}, e^{-t})$ is a solution to some linear system $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$. For which of the following initial conditions can you give the explicit solution of the linear system?

(a) $\mathbf{Y}(0) = (-2, 2)$ **(b)** $\mathbf{Y}(0) = (3, 4)$ **(c)** $\mathbf{Y}(0) = (0, 0)$ **(d)** $\mathbf{Y}(0) = (3, -3)$

34. The Linearity Principle is a fundamental property of systems of the form $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$. However, you should not assume that it is true for systems that are not of this form, no matter how simple. For example, consider the system

$$\frac{dx}{dt} = 1$$
$$\frac{dy}{dt} = x.$$

The following computations show that the Linearity Principle does not hold for this system.

(a) Show that $\mathbf{Y}(t) = (t, t^2/2)$ is a solution to this system.

(b) Show that $2\mathbf{Y}(t)$ is *not* a solution.

An Extended Linearity Principle that applies to systems such as this one is discussed in Chapter 4.

35. Given solutions $\mathbf{Y}_1(t) = (x_1(t), y_1(t))$ and $\mathbf{Y}_2(t) = (x_2(t), y_2(t))$ to the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}, \quad \text{where } \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we define the **Wronskian** of $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ to be the (scalar) function

$$W(t) = x_1(t)y_2(t) - x_2(t)y_1(t).$$

(a) Compute dW/dt.

(b) Use the fact that $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions of the linear system to show that

$$\frac{dW}{dt} = (a + d)\,W(t).$$

(c) Find the general solution of the differential equation $dW/dt = (a + d)\,W(t)$.

(d) Suppose that $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions to the system $d\mathbf{Y}/dt = \mathbf{AY}$. Verify that if $\mathbf{Y}_1(0)$ and $\mathbf{Y}_2(0)$ are linearly independent, then $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are also linearly independent for every t.

3.2 STRAIGHT-LINE SOLUTIONS

In Section 3.1 we discussed solutions of linear systems without worrying about how we came up with them (the rabbit-out-of-the-hat method). Often we used the time-honored method known as "guess and test." That is, we made a guess, then substituted the guess back into the equation and checked to see if it satisfied the system. However, the guess-and-test method is unsatisfying because it does not give us any understanding of where the formulas came from in the first place. In this section we use the geometry of the vector field to find special solutions of linear systems.

Geometry of Straight-Line Solutions

We begin by reconsidering an example from the previous section. The direction field for the linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}, \quad \text{where } \mathbf{A} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix},$$

is shown in Figure 3.7. Looking at the direction field, we see that there are two special lines through the origin. The first is the x-axis on which the vectors in the direction field all point directly away from the origin. The other special line runs from the second