EXERCISES FOR SECTION 2.7

1. For the SIR-model, show that $S(t) + I(t) + R(t) = 1$ for all t directly from the system of differential equations.

2. In the SIR model, we assume that everyone in the population is susceptible at time $t = 0$ except the very small fraction that is already infected. Suppose that some fraction of the population has received a vaccine, so they cannot catch the disease. The vaccine makes the fraction of the population that is susceptible at time $t = 0$ smaller.

(a) Using HPGSystemSolver applied to the SIR model with $\alpha = 0.25$ and $\beta = 0.1$, describe the behavior of the solutions with $I(0) = 0.01$ and $S(0) = 0.9, 0.8, 0.7, \dots$. Pay particular attention to the maximum of $I(t)$, that is, the maximum number of infecteds for each choice of $S(0)$. Also, note the limit of $S(t)$ as $t \to \infty$. (This limit is the fraction of the population that does not catch the disease during the epidemic.)

(b) If $\alpha = 0.25$ and $\beta = 0.1$, how large a fraction of the population must be vaccinated in order to keep the epidemic from getting started with $I(0) = 0.01$?

3. Vaccines make it possible to prevent epidemics. However, the time it takes to develop a vaccine may make it impossible to vaccinate everyone in a population before a disease arrives.

(a) For the SIR model, which initial conditions guarantee that $dI/dt < 0$? [*Hint*: Your answer should be expressed in terms of the parameters α and β.]

(b) For given values of α and β, what fraction of a population must be vaccinated before a disease arrives in order to prevent an epidemic?

4. In this section we showed that solution curves of the SIR model with $S(0) \approx 1$ and $I(0) \approx 0$ are graphs of the function

$$I(S) = -S + \frac{\beta}{\alpha}\ln(S) + 1.$$

(Note that the graph depends only on the ratio $\rho = \beta/\alpha$ of the parameters. Different values of the parameters can give the same value of ρ.)

(a) Determine the maximum value of $I(S)$ in terms of ρ.

(b) Is the statement "The epidemic cannot get started if $\beta > \alpha$" true or false? Justify your answer.

5. Let ρ denote the ratio β/α of the parameters α and β in the SIR model. Then

$$I(S) = -S + \rho\ln(S) + 1.$$

(a) Using graphing technology, graph $I(S)$ over the interval $0 < S \leq 1$ for various values of ρ between 0.1 and 1.0.

(b) Using the graphs that you produced in part (a), graph the solution of $I(S) = 0$ for $0 < S < 1$ as a function of ρ.

(c) What does the graph that you produced in part (b) tell you about the long-term predictions of the SIR model in terms of the ratio ρ?

6. One of the basic assumptions of the SIR model is that individuals who recover from the disease never get it again. However, diseases continually evolve, and new strains can emerge that can infect those who have recovered from the previous strain. In this exercise, we modify the SIR model so that recovereds become susceptible again in a linear rate. We obtain the system of equations

$$\frac{dS}{dt} = -\alpha SI + \gamma R$$
$$\frac{dI}{dt} = \alpha SI - \beta I$$
$$\frac{dR}{dt} = \beta I - \gamma R$$

(a) Show that the sum $S(t) + I(t) + R(t)$ is constant as a function of t for this model.

(b) Derive a system in the two dependent variables S and I using the fact that $R = 1 - (S + I)$.

(c) What are the equilibrium points for this model of the two variables S and I? (*Hint*: Both S and I are nonnegative, and $S(t) + I(t) \leq 1$ for all t.)

(d) Fix $\alpha = 0.3$, $\beta = 0.15$, and $\gamma = 0.05$ and use HPGSystemSolver to sketch the phase portrait. Describe the behavior of solutions.

(e) How does the system change if we fix $\alpha = 0.3$ and $\beta = 0.15$, but vary γ over a small interval surrounding $\gamma = 0.05$?

7. In the movie *I Am Legend*, the infecteds work together to increase the number of infecteds. We can modify the SIR model to include the assumption that zombies actively infect susceptibles by replacing I by $\sqrt{I}$ in the interaction term. (Note that $0 \leq I \leq 1$, so $\sqrt{I} \geq I$.) We obtain the system

$$\frac{dS}{dt} = -\alpha S\sqrt{I}$$
$$\frac{dI}{dt} = \alpha S\sqrt{I} - \beta I.$$

(a) Calculate the equilibrium points of this model.

(b) Find the region of the phase plane where $dI/dt > 0$.

(c) Use $\alpha = 0.2$ and $\beta = 0.1$ and sketch the phase portrait. What does the model predict for the spread of the zombies in this case?

8. Many zombie movies are based on the premise that zombies do not stop infecting new victims until they are destroyed by a susceptible. In addition, the susceptibles destroy as many zombies as they can. We can model the spread of zombies in such a movie by assuming that infecteds (zombies) become recovereds (zombies who can not infect susceptibles) at a rate proportional to the size of the remaining susceptible population. We obtain the system

$$\frac{dS}{dt} = -\alpha SI$$
$$\frac{dI}{dt} = \alpha SI - \gamma S.$$

(a) Calculate the equilibrium points of this model.
(b) Find the region of the phase plane where $dI/dt > 0$.
(c) Use $\alpha = 0.2$ and $\gamma = 0.1$ and sketch the phase portrait. What does the model predict for the spread of the zombies in this case?

The SIR model is particularly relevant to a homogenous population in an enviroment with little geographic distribution. A famous example of exactly this situation occurred in 1978 at a British boarding school.* A single boy in the school of 763 students contracted the flu and the epidemic spread rapidly, as shown in Table 2.3. (We are assuming that the number of students confined to bed was the same as the number of infected students.)

Table 2.3
The daily count of the number of infected students.

t	Infected	t	Infected	t	Infected
0	1	5	222	10	123
1	3	6	282	11	70
2	7	7	256	12	25
3	25	8	233	13	11
4	72	9	189	14	4

9. Assume that the parameter $\alpha = 1.66$ in the SIR model for the data in Table 2.3.

(a) Using whatever technology that is most convenient, determine an appropriate value of β that matches the data in Table 2.3.
(b) Using the value of β that you computed in part (a), calculate the total number of students who caught the flu during the epidemic.
(c) Interpret the value of β that you computed in part (a) in terms of the length of time that students with the flu remained infected.

10. Using $\alpha = 1.66$ and the value of β that you determined in Exercise 9, how would the progress of the epidemic have changed if 200 students had been vaccinated before the disease started? (Give as precise an answer as possible.)

*Anonymous, "Epidemiology: Influenza in a boarding school," *British Medical Journal*, Vol. 4, 1978, p. 587.