EXERCISES FOR SECTION 1.9

In Exercises 1–6, find the general solution of the differential equation specified.

1. $\dfrac{dy}{dt} = -\dfrac{y}{t} + 2$

2. $\dfrac{dy}{dt} = \dfrac{3}{t}y + t^5$

3. $\dfrac{dy}{dt} = -\dfrac{y}{1+t} + t^2$

4. $\dfrac{dy}{dt} = -2ty + 4e^{-t^2}$

5. $\dfrac{dy}{dt} - \dfrac{2t}{1+t^2}y = 3$

6. $\dfrac{dy}{dt} - \dfrac{2}{t}y = t^3 e^t$

In Exercises 7–12, solve the given initial-value problem.

7. $\dfrac{dy}{dt} = -\dfrac{y}{1+t} + 2, \quad y(0) = 3$

8. $\dfrac{dy}{dt} = \dfrac{1}{t+1}y + 4t^2 + 4t, \quad y(1) = 10$

9. $\dfrac{dy}{dt} = -\dfrac{y}{t} + 2, \quad y(1) = 3$

10. $\dfrac{dy}{dt} = -2ty + 4e^{-t^2}, \quad y(0) = 3$

11. $\dfrac{dy}{dt} - \dfrac{2y}{t} = 2t^2, \quad y(-2) = 4$

12. $\dfrac{dy}{dt} - \dfrac{3}{t}y = 2t^3 e^{2t}, \quad y(1) = 0$

In Exercises 13–18, the differential equation is linear, and in theory, we can find its general solution using the method of integrating factors. However, since this method involves computing two integrals, in practice it is frequently impossible to reach a formula for the solution that is free of integrals. For these exercises, determine the general solution to the equation and express it with as few integrals as possible.

13. $\dfrac{dy}{dt} = (\sin t)y + 4$

14. $\dfrac{dy}{dt} = t^2 y + 4$

15. $\dfrac{dy}{dt} = \dfrac{y}{t^2} + 4\cos t$

16. $\dfrac{dy}{dt} = y + 4\cos t^2$

17. $\dfrac{dy}{dt} = -\dfrac{y}{e^{t^2}} + \cos t$

18. $\dfrac{dy}{dt} = \dfrac{y}{\sqrt{t^3 - 3}} + t$

19. For what value(s) of the parameter a is it possible to find explicit formulas (without integrals) for the solutions to

$$\frac{dy}{dt} = aty + 4e^{-t^2}?$$

20. For what value(s) of the parameter r is it possible to find explicit formulas (without integrals) for the solutions to

$$\frac{dy}{dt} = t^r y + 4?$$

21. Consider the nonhomogeneous equation

$$\frac{dv}{dt} + 0.4v = 3\cos 2t.$$

(a) Find the general solution using the method of integrating factors.

(b) Find the general solution using the guessing technique from Section 1.8.

Comment on which method was easier for you.

22. In this exercise, we explore the connections between the method of integrating factors discussed in this section and the Extended Linearity Principle. Consider the nonhomogeneous linear equation

$$\frac{dy}{dt} = a(t)y + b(t),$$

where $a(t)$ and $b(t)$ are continuous for all t.

(a) Let

$$\mu(t) = e^{-\int_0^t a(\tau)\,d\tau}.$$

Show that $\mu(t)$ is an integrating factor for the nonhomogeneous equation.

(b) Show that $1/\mu(t)$ is a solution to the associated homogeneous equation.

(c) Show that

$$y_p(t) = \frac{1}{\mu(t)} \int_0^t \mu(\tau)\, b(\tau)\, d\tau$$

is a solution to the nonhomogeneous equation.

(d) Use the Extended Linearity Principle to find the general solution of the nonhomogeneous equation.

(e) Compare your result in part (d) to the formula

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)\, b(t)\, dt$$

for the general solution that we obtained on page 126.

(f) Illustrate the calculations that you did in this exercise for the example

$$\frac{dy}{dt} = -2ty + 4e^{-t^2}.$$

23. Consider the nonhomogeneous equation

$$\frac{dy}{dt} + 2y = 3e^{-2t}.$$

In Section 1.8, we saw that the guess $y_p(t) = \alpha e^{-2t}$ does not produce a solution because it is a solution to the associated homogeneous equation. We then guessed $y_p(t) = \alpha t e^{-2t}$. Use the method of integrating factors to explain why this guess is a good idea.

24. A 30-gallon tank initially contains 15 gallons of salt water containing 6 pounds of salt. Suppose salt water containing 1 pound of salt per gallon is pumped into the top of the tank at the rate of 2 gallons per minute, while a well-mixed solution leaves the bottom of the tank at a rate of 1 gallon per minute. How much salt is in the tank when the tank is full?

25. A 400-gallon tank initially contains 200 gallons of water containing 2 parts per billion by weight of dioxin, an extremely potent carcinogen. Suppose water containing 5 parts per billion of dioxin flows into the top of the tank at a rate of 4 gallons per minute. The water in the tank is kept well mixed, and 2 gallons per minute are removed from the bottom of the tank. How much dioxin is in the tank when the tank is full?

26. A 100-gallon tank initially contains 100 gallons of sugar water at a concentration of 0.25 pounds of sugar per gallon. Suppose that sugar is added to the tank at a rate of p pounds per minute, that sugar water is removed at a rate of 1 gallon per minute, and that the water in the tank is kept well mixed.

- **(a)** What value of p should we pick so that, when 5 gallons of sugar solution is left in the tank, the concentration is 0.5 pounds of sugar per gallon?
- **(b)** Is it possible to choose p so that the last drop of water out of the bucket has a concentration of 0.75 pounds of sugar per gallon?

27. Suppose a 50-gallon tank contains a volume V_0 of clean water at time $t = 0$. At time $t = 0$, we begin dumping 2 gallons per minute of salt solution containing 0.25 pounds of salt per gallon into the tank. Also at time $t = 0$, we begin removing 1 gallon per minute of salt water from the tank. As usual, suppose the water in the tank is well mixed so that the salt concentration at any given time is constant throughout the tank.

- **(a)** Set up the initial-value problem for the amount of salt in the tank. [*Hint*: The initial value of V_0 will appear in the differential equation.]
- **(b)** What is your equation if $V_0 = 0$ (the tank is initially empty)? Comment on the validity of the model in this situation. What will be the amount of salt in the tank at time t for this situation?