a resistor in parallel with the other resistor. Let R_L be the resistance that is due solely to the lamp, then it can be shown that the differential equation that governs v_c over the time interval $T_O < t < T_O + T_C$ is

$$\frac{dv_c}{dt} = \frac{K}{RC} - \left(\frac{R + R_L}{RR_L C}\right) v_c.$$

Note that increased resistance due to the lamp causes v_c to decrease faster than it increased when the switch was open.

The light switch remains closed over the interval $T_O < t < T_O + T_C$, and we pick T_C so that $v_c(T_O + T_C) = v_0$. In other words, we pick T_C so that the voltage v_c is periodic with period $T_O + T_C$ (see Figure 1.30). For this example, the slope field is discontinuous along infinitely many vertical lines, that is, the lines $t = T_O$, $t = T_O + T_C$, $t = 2T_O + T_C$, $t = 2(T_O + T_C)$, ... (see Figure 1.30).

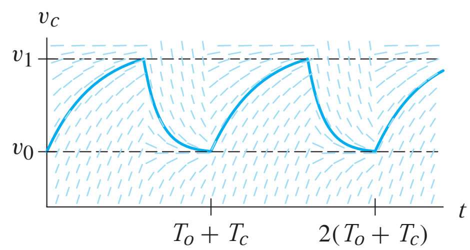


Figure 1.30
The graph of the solution and its corresponding slope field for the flashing light example in the case where $R = 0.5$, $R_L = 0.25$, $C = 1$, and $K = 2.5$. In this case, we have made $T_O = 1$ and $T_C = 0.7$, so the solution is periodic with period 1.7.

Combining Qualitative with Quantitative Results

When only knowledge of the qualitative behavior of the solution is required, sketches of solutions obtained from slope fields can sometimes suffice. In other applications it is necessary to know the exact value (or almost exact value) of the solution with a given initial condition. In these situations analytic and/or numerical methods can't be avoided. But even then, it is nice to have graphs of the solutions.

EXERCISES FOR SECTION 1.3

In Exercises 1–6, sketch the slope fields for the differential equation as follows:

(a) Pick a few points (t, y) with both $-2 \le t \le 2$ and $-2 \le y \le 2$ and plot the associated slope marks without the use of technology.

(b) Use HPGSolver to check these individual slope marks.

(c) Make a more detailed drawing of the slope field and then use HPGSolver to confirm your answer.

For more details about HPGSolver and other programs that are part of the DETools package, see the description of DETools inside the front cover of this book.

1. $\dfrac{dy}{dt} = t^2 + t$ **2.** $\dfrac{dy}{dt} = t^2 + 1$ **3.** $\dfrac{dy}{dt} = 1 - 2y$

4. $\dfrac{dy}{dt} = 4y^2$ **5.** $\dfrac{dy}{dt} = 2y(1 - y)$ **6.** $\dfrac{dy}{dt} = y + t + 1$

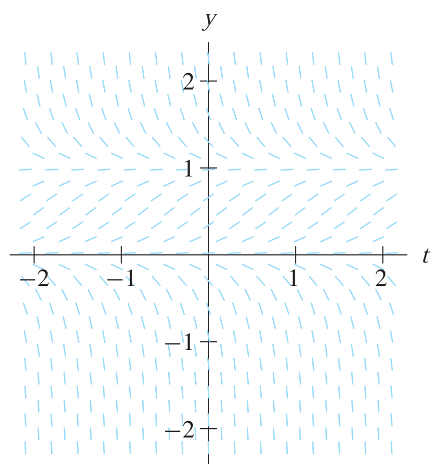
In Exercises 7–10, a differential equation and its associated slope field are given. For each equation,

(a) sketch a number of different solutions on the slope field, and

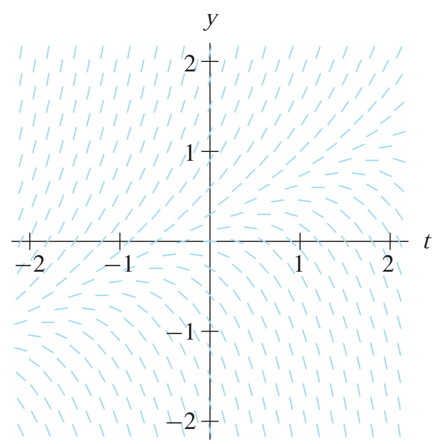
(b) describe briefly the behavior of the solution with $y(0) = 1/2$ as t increases.

You should first answer these exercises without using any technology, and then you should confirm your answer using `HPGSolver`.

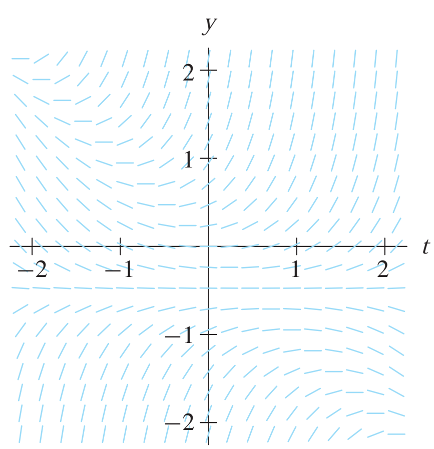
7. $\dfrac{dy}{dt} = 3y(1 - y)$



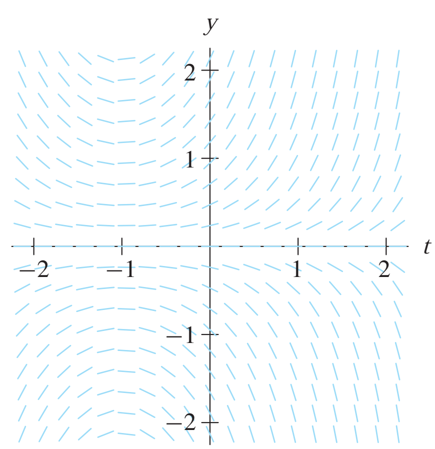
8. $\dfrac{dy}{dt} = 2y - t$



9. $\dfrac{dy}{dt} = \left(y + \frac{1}{2}\right)(y + t)$



10. $\dfrac{dy}{dt} = (t + 1)y$



11. Suppose we know that the function $f(t, y)$ is continuous and that $f(t, 3) = -1$ for all t.

(a) What does this information tell us about the slope field for the differential equation $dy/dt = f(t, y)$?

(b) What can we conclude about solutions $y(t)$ of $dy/dt = f(t, y)$? For example, if $y(0) < 3$, can $y(t) \to \infty$ as t increases?

12. Suppose the constant function $y(t) = 2$ for all t is a solution of the differential equation

$$\frac{dy}{dt} = f(t, y).$$

(a) What does this tell you about the function $f(t, y)$?

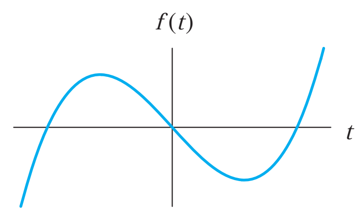
(b) What does this tell you about the slope field? In other words, how much of the slope field can you sketch using this information?

(c) What does this tell you about solutions with initial conditions $y(0) \neq 2$?

13. Suppose we know that the graph to the right is the graph of the right-hand side $f(t)$ of the differential equation

$$\frac{dy}{dt} = f(t).$$

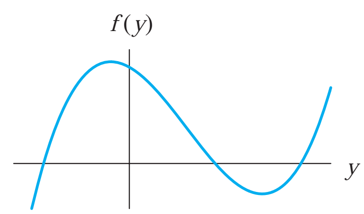
Give a rough sketch of the slope field that corresponds to this differential equation.



14. Suppose we know that the graph to the right is the graph of the right-hand side $f(y)$ of the differential equation

$$\frac{dy}{dt} = f(y).$$

Give a rough sketch of the slope field that corresponds to this differential equation.



15. Consider the autonomous differential equation

$$\frac{dS}{dt} = S^3 - 2S^2 + S.$$

(a) Make a rough sketch of the slope field without using any technology.

(b) Using this drawing, sketch the graphs of the solutions $S(t)$ with the initial conditions $S(0) = 1/2$, $S(1) = 1/2$, $S(0) = 1$, $S(0) = 3/2$, and $S(0) = -1/2$.

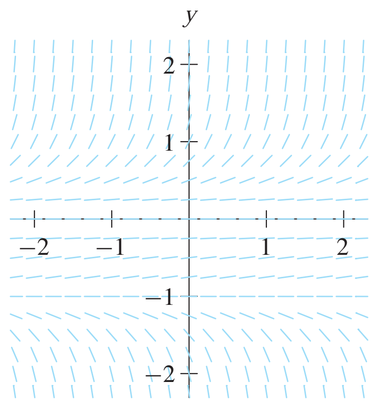
(c) Confirm your answer using HPGSolver.

16. Eight differential equations and four slope fields are given below. Determine the equation that corresponds to each slope field and state briefly how you know your choice is correct. You should do this exercise without using technology.

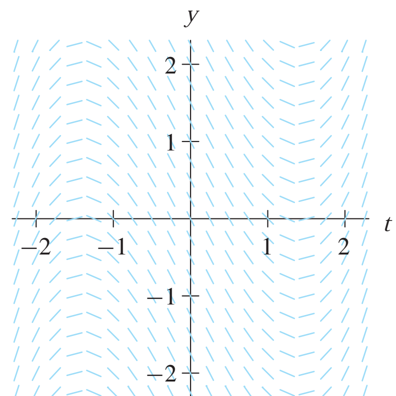
(i) $\frac{dy}{dt} = y^2 + y$ (ii) $\frac{dy}{dt} = y^2 - y$ (iii) $\frac{dy}{dt} = y^3 + y^2$ (iv) $\frac{dy}{dt} = 2 - t^2$

(v) $\frac{dy}{dt} = ty + ty^2$ (vi) $\frac{dy}{dt} = t^2 + t^2 y$ (vii) $\frac{dy}{dt} = t + ty$ (viii) $\frac{dy}{dt} = t^2 - 2$

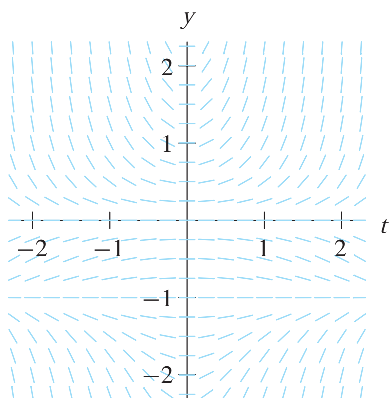
(a)



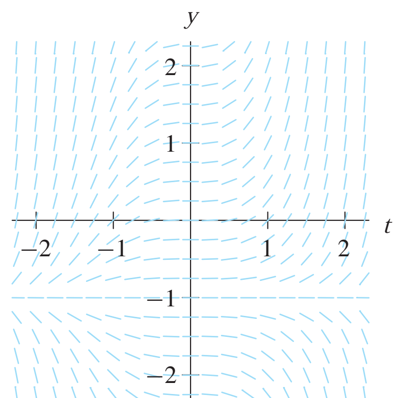
(b)



(c)



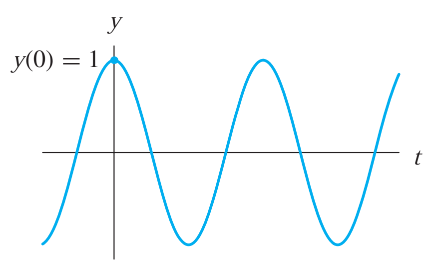
(d)



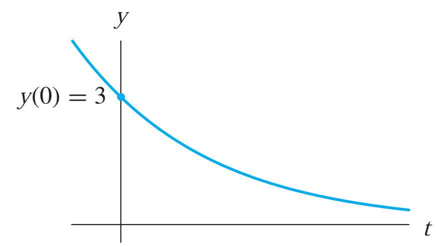
17. Suppose we know that the graph below is the graph of a solution to $dy/dt = f(t)$.

(a) How much of the slope field can you sketch from this information? [*Hint*: Note that the differential equation depends only on t.]

(b) What can you say about the solution with $y(0) = 2$? (For example, can you sketch the graph of this solution?)



18. Suppose we know that the graph below is the graph of a solution to $dy/dt = f(y)$.



(a) How much of the slope field can you sketch from this information? [*Hint*: Note that the equation is autonomous.]

(b) What can you say about the solution with $y(0) = 2$? Sketch this solution.

19. The spiking of a neuron can be modeled* by the differential equation

$$\frac{d\theta}{dt} = 1 - \cos\theta + (1 + \cos\theta)I(t),$$

where $I(t)$ is the input. Often the input function $I(t)$ is a constant I. When θ is an odd multiple of π, the neuron spikes.

(a) Using HPGSolver, sketch three slope fields, one for each of the following values of I: $I_1 = -0.1$, $I_2 = 0.0$, and $I_3 = 0.1$.

(b) Calculate the equilbrium solutions for each of these three values.

(c) Using the slope field, describe the long-term behavior of the solutions in each of the three cases.

20. By separating variables, find the general solution of the differential equation

$$\frac{dv_c}{dt} = -\frac{v_c}{RC},$$

where R and C are constants. Then check your answer by substituting it back into the differential equation.

21. By separating variables, find the general solution of the differential equation

$$\frac{dv_c}{dt} = \frac{K - v_c}{RC},$$

where R, C, and K are constants. Then check your answer by substituting it back into the differential equation.

22. By separating variables, find the solution of the initial-value problem

$$\frac{dv_c}{dt} = \frac{V(t) - v_c}{RC}, \quad v_c(0) = 6$$

where $R = 0.5$, $C = 1.0$, and $V(t)$ is the function that is constantly 3 for $t < 3$ and 0 for $t > 3$. Then check your answer by substituting it back into the differential equation. [*Hint*: Do this exercise in two steps. First, solve the equation for $t \leq 3$. Then use the value $v_c(3)$ to state another initial-value problem.]

*This model is often referred to as the theta model, but it is also called the Ermentrout-Kopell canonical model. See "Parabolic bursting in an excitable system coupled with a slow oscillation" by G. B. Ermentrout and N. Kopell, in *SIAM J. Applied Math*, Vol. 44, 1984, pp. 1133–1149.