Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

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Existence and uniqueness

Existence and uniqueness theorem

(Picard - Lindelöff)

$$\frac{dx}{dt} = f(x,t) \iff x(t) = x_0 + \int_{t_0}^{t} f(x(s),s)\,ds$$

$$x(t_0) = x_0$$

$$T(x) = x_0 + \int_{t_0}^{t} f(x(s),s)\,ds$$

Statement of the theorem

Existence and uniqueness theorem (Chicone, dependence on initial conditions)

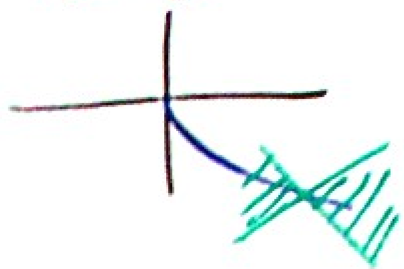
$\frac{dx}{dt} = f(x,t)$

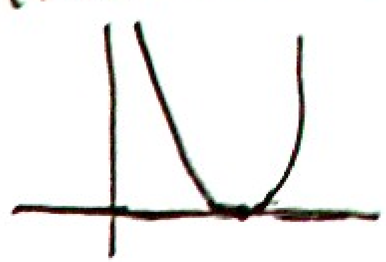
Thm Suppose that $f : \Omega \times J \to \mathbb{R}^n$ where $J \subseteq \mathbb{R}$ & $\Omega \subseteq \mathbb{R}^n$, f is continuous & Lipschitz with respect to $x \in \Omega$. For every $(x_0, t_0) \in \Omega \times J$, we can find open intervals $J_0 \subseteq J$, $\Omega_0 \subseteq \Omega$ such that there is a unique solution $x(t, x_0)$ to the initial value problem defined $\forall t \in J_0$ with $x(t_0, x_0) = x_0$.

The soln is continuous in $J_0 \times \Omega_0$.

Globally.
f Lipschitz
$x, y \in \Omega$
$\exists L > 0$ s.t.
$|f(x) - f(y)| \leq L|x - y|$

$\dot{x} = -\sqrt{x}$





Proof. Part one

Existence and uniqueness theorem (Chicone, dependence on initial conditions)

$\frac{dx}{dt} = f(x,t)$

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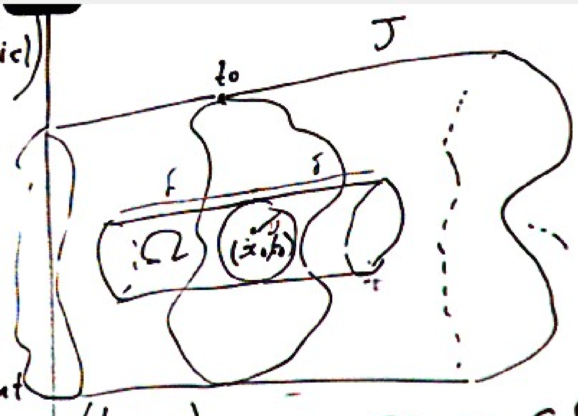
The soln is continuous in $J_0 \times \Omega_0$.

Pf

$$\text{IVP}\begin{cases} \frac{dx}{dt} = f(x,t) \\ x(t_0, x_0) = x_0 \end{cases} \iff x(x_0, t) = x_0 + \int_{t_0}^{t} f(x(s), s)\, ds$$

Let $B(t_0, \delta)$ & $B(x_0, \nu)$ two balls of radius δ and ν, with δ and ν sufficiently small so that $B(t_0,\delta) \times B(x_0, \nu) \subset J \times \Omega$.



The properties of $f \Rightarrow \exists\, M, L$ such that

$$\sup_{\substack{t \in B(t_0,\delta) \\ x \in B(x_0,\nu)}} |f(x,t)| \le M \quad \text{(cont)}$$

(Lip) $\Rightarrow$ $x_1, x_2 \in \Omega$, $t \in J$ $|f(x_1,t) - f(x_2,t)| \le L\,|x_1 - x_2|$

We choose δ such that

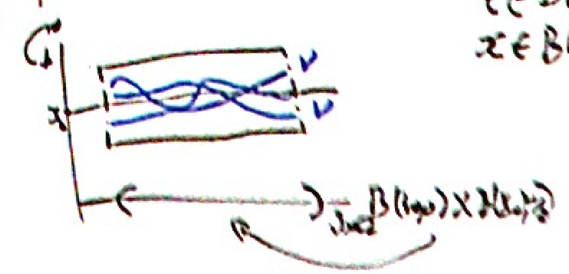
$$\delta L < \min\{1, \nu/2\}$$

$$\delta M < \nu/2$$

Let $C_b^0 = C_b^0(B(x_0, \nu/2) \times B(t_0, \delta), \mathbb{R}^n)$

Define

$$X = \{\phi \in C_b^0 : \|\phi - x_0\| := \sup_{\substack{t \in B(t_0,\delta) \\ x \in B(x_0,\nu/2)}} |\phi(x,t) - x_0| \le \nu\}$$



Proof. Part two.

Existence and uniqueness theorem (Chicone, dependence on initial conditions)

$\frac{dx}{dt} = f(x,t)$

Thm Suppose that $f : \Omega \times J \to \mathbb{R}^n$ where $J \subseteq \mathbb{R}$ & $\Omega \subseteq \mathbb{R}^n$, f is continuous & Lipschitz with respect to $x \in \Omega$. For every $(x_0, t_0) \in \Omega \times J$, we can find open intervals $J_0 \subseteq J$, $\Omega_0 \subseteq \Omega$ such that there is a unique solution $x(t, x_0)$ to the initial Value problem defined $\forall t \in J_0$ with $x(t_0, x_0) = x_0$.

The soln is continuous in $J_0 \times \Omega_0$.

Pf (cont)

The operator

$$\Lambda(\phi)(x,t) := x + \int_{t_0}^{t} f(\phi(x,s),s)\, ds. \quad (\in C_b^0)$$

First we prove that $\Lambda : \mathbb{X} \to \mathbb{X}$

than we prove that the contracting constent is < 1.

$\Lambda(\phi)$ is continuous both in x & t and

$$|\Lambda(\phi)(x,t) - x_0| = \left| x - x_0 + \int_{t_0}^{t} f(\phi(x,s),s)\, ds \right|$$

$$\leq |x - x_0| + \int_{t_0}^{t} |f(\phi(x,s),s)|\, ds$$

$$\leq |x - x_0| + M(t - t_0) < \nu/2 + M\delta \leq \nu/2 + \nu/2 = \nu$$

$$\Rightarrow \Lambda(\phi) \in \mathbb{X}$$

$\phi_1, \phi_2 \in \mathbb{X}$

$$\|\Lambda(\phi_1) - \Lambda(\phi_2)\| = \sup_{\substack{t \in B(t_0,\delta) \\ x \in B(x_0,\nu/2)}} |\Lambda(\phi_1)(x,t) - \Lambda(\phi_2)(x,t)|$$

$$\leq \sup_{\substack{t \in B(t_0,\delta) \\ x \in B(x_0,\nu/2)}} \int_{t_0}^{t} |f(\phi_1(x,s),s) - f(\phi_2(x,s),s)|\, ds$$

$$\leq \sup_{\substack{t \in B(t_0,\delta) \\ x \in B(x_0,\nu/2)}} L \int_{t_0}^{t} |\phi_1(x,s) - \phi_2(x,s)|\, ds = \sup_{\substack{t \in B(t_0,\delta) \\ x \in B(x_0,\nu/2)}} L \|\phi_1 - \phi_2\| (t - t_0)$$

$$\leq L\delta \|\phi_1 - \phi_2\|$$

So Λ is a contraction and there is a fixed point $\Lambda(\phi^*) = \phi^*$ and its unique!

$$\phi^*(x,t) = x + \int_{t_0}^{t} f(\phi^*(x,s),s)\, ds$$

It's the unique solution satisfying the IVP.

Comments

Existence and uniqueness the[orem]

$\frac{dx}{dt} = f(x,t)$ where $f : \Omega \times J \to \mathbb{R}^n$

Thm Suppose that $J \subseteq \mathbb{R}$ & $\Omega \subseteq \mathbb{R}^n$, f is continuous & Lipschitz with respect to $x \in \Omega$. For every $(x_0, t_0) \in \Omega \times J$, we can find open intervals $J_0 \subseteq J$, $\Omega_0 \subseteq \Omega$ such that there is a unique solution $x(t, x_0)$ to the initial value problem defined $\forall t \in J_0$ with $x(t_0, x_0) = x_0$. The soln is continuous in $J_0 \times \Omega_0$.

Comments

The size of the interval $J_0 = B(t_0, \delta)$ could be extremely small.

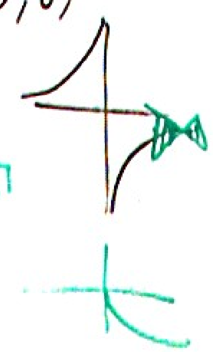
$\delta L < \min\{1, 1/2\}$

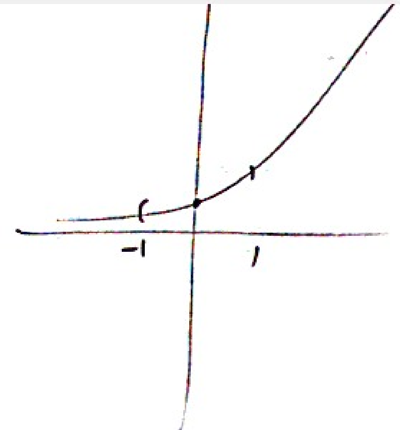
$\delta M < 1/2$

$\dot{x} = -\frac{1}{x}$

$\dot{x} = -\sqrt{x}$

$\dot{x} = x$





Moral of the story is that the size of the interval J_0 depends heavily on the Lipschitz constant.

We know that there are examples for which the solution exists for larger size intervals.

What to do? L

This is how we extend the solution.

Global uniqueness in time.

Global uniqueness in time

Thm
Under the hypotheses of the ∃! thm, the solutions are unique for all the times for which they exist

Pf
$x(t)$, $y(t)$ two solutions to $\frac{d\phi}{dt} = f(\phi, t)$
with $x(t_0) = x_0$, $y(t_0) = y_0$

$$|x(t) - y(t)| \leq |x_0 - y_0| + \int_{t_0}^{t} |f(x(s), s) - f(y(s), s)| ds$$
$$\leq |x_0 - y_0| + L \int_{t_0}^{t} |x(s) - y(s)| ds$$

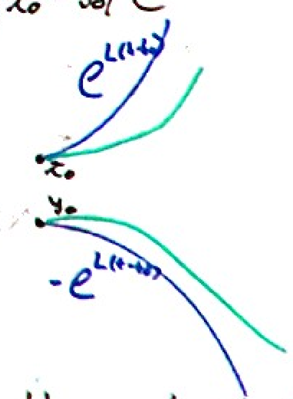
We define the function $v(t) = |x(t) - y(t)|$

$$v(t) \leq \underbrace{|x_0 - y_0|}_{v(t_0)} + \int_{t_0}^{t} L\, v(s) ds$$

$\phi = v$
$\alpha = |x_0 - y_0|$
$\psi = L$

Gronwall's inequality

$$v(t) \leq |x_0 - y_0| e^{L(t-t_0)}$$
$$\Rightarrow |x(t) - y(t)| \leq |x_0 - y_0| e^{L(t-t_0)}$$



If $x_0 = y_0$ then $|x(t) - y(t)| = 0$ and the solutions are the same and therefore unique for all the times in which they exist. //

The maximal interval of existence is the maximal interval for which we can extend the solution.

Regularity

Regularity

$$\frac{dx}{dt} = f(x,t) \iff x(t) = x_0 + \int_{t_0}^{t} f(x(s),s)\,ds$$

$x(t_0) = x_0$

What is the regularity of the solution?

The function is C^1

Now if $f \in C^r$ (in the first argument)

then $x \in C^{r+1}$

if $f \in C^\infty$

$x \in C^\infty$

Think over the weekend.

Is x analytic whenever f is analytic?