Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

Renato Calleja, 7 de marzo de 2024

• φ_t flujo de $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, f al menos C^1

$\Rightarrow$ $\underline{D_\xi \varphi_t(\xi)}$ es m.f.s.p. de

$$\dot{W} = Df(\varphi_t(\xi))W$$
$$W(0) = Id.$$

$d(\varphi_t)_\xi$, sus columnas son $\frac{\partial}{\partial \xi_i}(\varphi_t)(\xi)$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \xi_i}\varphi_t(\xi)\right) = \frac{\partial}{\partial \xi_i}\left(\frac{d}{dt}\varphi_t(\xi)\right) = \frac{\partial}{\partial \xi_i}\left(f(\varphi_t(\xi))\right)$$

$$= df_{\varphi_t(\xi)}\left(\frac{\partial}{\partial \xi_i}\varphi_t(\xi)\right)$$

$$d(\underline{\varphi_0})_\xi = d(Id)_\xi = Id.$$

Lyapunov exponents

$$\chi(\varphi, v) := \limsup_{t\to\infty} \frac{1}{t}\log\left(\frac{|D\varphi_t(\xi)v|}{|v|}\right)$$

$$\dot{x} = A(t)x, \quad A(t+T) = A(t), \quad \forall t \in \mathbb{R} \quad \ldots (1)$$

Prop

If μ is a Floquet exponent of the system (1), then the real part of μ is the Lyapunov exp.

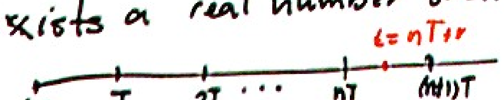
Proof

A fundamental solution matrix of (1) is $\Phi(t)$

$\exists v \in \mathbb{R}^n$ s.t. $\Phi(T)v = e^{\mu T}v$

For $t \geq 0$ there always exists a real number such that

$t = nT + r$, $n \in \mathbb{N}$, $r \in (0, T)$



Floquet normal form.

$$D\Phi(nT+r)v = P(nT+r)\, e^{(nT+r)B}\, v$$

Now

$$\frac{1}{t}\log\left(\frac{|D\Phi(t)v|}{|v|}\right) = \frac{1}{nT+r}\log\left(\frac{|D\Phi(nT+r)v|}{|v|}\right)$$

$$= \frac{1}{nT+r}\log\left(\frac{|P(r)\, e^{r\mu} e^{nT\mu} v|}{|v|}\right)$$

$$= \frac{1}{T}\left(\frac{nT}{nT+r}\right)\left(\frac{1}{n}\log\left(\frac{|P(r)\, e^{r\mu} e^{nT\mu} v|}{|v|}\right)\right)$$

$$= \frac{1}{T}\left(\frac{nT}{nT+r}\right)\left(\frac{1}{n}\left[\log\left(|e^{nT\mu}|\right) + \log\left(\frac{|P(r)\, e^{r\mu} v|}{|v|}\right)\right]\right)$$

$$= \frac{1}{T}\left(\frac{nT}{nT+r}\right)\left(\frac{1}{n}\log\left(e^{nT\mathrm{Re}(\mu)}\right) + \frac{1}{n}\log\left(\frac{|P(r)e^{r\mu}v|}{|v|}\right)\right)$$

$$= \left(\frac{nT}{nT+r}\right)\frac{1}{nT}\, nT\,\mathrm{Re}(\mu) + O(1/n) \xrightarrow[n\to\infty]{} \mathrm{Re}(\mu) = \chi$$

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What about non-linear systems?

Linearization and local stability.

What is the meaning of "local" here.

We can obtain information of the system in a neighborhood of a special solution.

We use smoothness of the system.

Let's talk about a fixed point (simple)

$$\dot{x} = f(x), \quad f: \mathbb{R}^n \to \mathbb{R}^n \qquad (2)$$

and $f(p) = 0 \leftarrow$ fixed point.

$x_0(t) = p$ this is a solution to (2) $\forall t \in \mathbb{R}$.

Assume that f is smooth in a nbhd of p.

then we use Taylor.

We look for solutions

$x = p + y, \quad |y| < \delta \quad \delta$ is small.

Then

$$f(x) = f(p+y) = \cancel{f(p)}^{0} + Df(p)y + \mathcal{O}(|y|^2)$$

Now think of solutions in this nbhd.

$x(t) = p + y(t)$

For y we have that

$\dot{y} = Df(p)y + N(y)$

$N(y) = f(p+y) - Df(p)y \underset{\text{Taylor.}}{=} \mathcal{O}(|y|^2)$

Note that

$Df: \mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) = \mathbb{R}^{n \times n}$

$\hookrightarrow$ matrix of the partial derivatives of f.

$$\to \lim_{\substack{h \to 0 \\ h \in \mathbb{R}^n}} \frac{1}{|h|} \left| f(x_0+h) - f(x_0) - Df(x_0)h \right|_{\mathbb{R}^n} = 0$$

$D^2 f: \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n))$

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}^n}} \frac{1}{|h|} \| Df(x_0+h) - Df(x_0) - D^2 f(x_0)h \|_{\mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)} = 0$$

$D^2 f$ is the Hessian of f

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$$\boxed{\begin{array}{l} f(x,t) \\ \dot{y} = A(t)y + g(y,t) \end{array}}$$

$$N(y) = f(p+y) - Df(p)y \underset{\text{Taylor.}}{=} \mathcal{O}(|y|^2)$$

Compare to

$$\dot{x} = A(t)x + \underbrace{N(t)}_{\text{forcing term.}} \xrightarrow{\text{solution}} \underbrace{x(t) = \Phi(t)x(t_0) + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)N}_{\text{Duhamel's formula}}$$

$$\dot{x}(t) = \dot{\Phi}(t)x(t_0) + \dot{\Phi}(t)\int_{t_0}^{t}\Phi^{-1}(s)N(s)ds + \Phi(t)\left[\Phi^{-1}(t)N(t) - \Phi^{-1}(t_0)\right.$$

$$= A(t)\Phi(t)\left[x(t_0) + \int_{t_0}^{t}\Phi^{-1}(s)N(s)ds\right] + N(t)$$

$$= A(t)x(t) + N(t)$$ so Duhamel's formula satisfies the equation.

LIVE 100%

What about non-linear systems?

Lyapunov stability

We say that a solution X_0 is Lyapunov stable if all the solutions that start near x_0, converge to x_0 as $t \to \infty$. (x_0 is asymtotically stable).

Thm

Consider the initial value problem (5)

$\dot{x} = Ax + g(x,t)$, $x(t_0) = x_0$

If all the eigenvalues of A have negative real part, and there are constants $a > 0$ and $k > 0$ such that $\|g(x,t)\| \le k\|x\|^2$ whenever $\|X\| < a$, then there exist positive constants c, b & α that are independent of t_0 such that the soln of (5) satisfy that

$\|x(t)\| \le c\|x_0\| e^{-\alpha(t-t_0)}$, for $t \geq t_0$ if $\|x_0\| \le b$. In particular, the function $t \mapsto x(t)$ is defined $\forall\, t \geq t_0$ and the zero solution is asymtotically stable.

We know that $g(0,t) = 0$

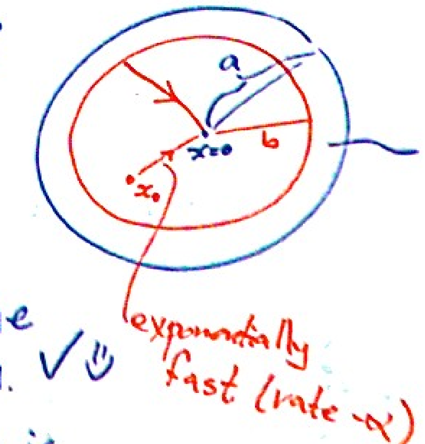
0 is a solution for every $t \geq t_0$

$\dot{x} = Ax$
Globally stable

$$\boxed{\begin{array}{l} f(X,t) \\ \dot{y} = A(t)y + g(y,t) \end{array}} \quad (4)$$

Comes from the smoothness of the v.f.

$\|g(X,t)\| \le k\|X\|^2$



The upside is that you recover asyptotic stability from the linear problem. ✓

The down side is that it is a local result.

locally stable.