Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

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Examples

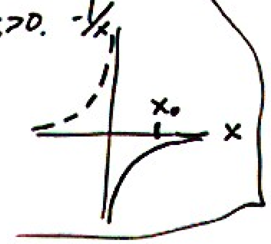
Recap

$\dot{x} = x^2$, $\phi(t, x_0) = \dfrac{x_0}{1 - x_0 t}$, $|\phi(t, x_0)| \xrightarrow[t \to x_0^{-1}]{} \infty$

Example 2

$\dot{x} = -\dfrac{1}{x}$, The function is only defined for $x > 0$. $\phi(0, x_0) = x_0 > 0$

$J = \mathbb{R}$, $\Omega = \{x > 0\}$



$$\left(\frac{d}{dt}\phi(t, x_0)\right)\phi(t, x_0) = -1$$

$$\frac{d}{dt}\left(\frac{1}{2}\phi^2(t, x_0)\right) = -1$$

$$\int_0^t \frac{d}{ds}\left(\frac{1}{2}\phi^2(s, x_0)\right) ds = -t$$

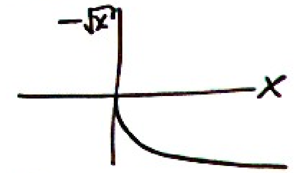
$$\phi^2(t, x_0) - \underbrace{\phi^2(0, x_0)}_{x_0^2} = -2t \qquad J_0 = (-\infty, x_0^2/2)$$

$$\phi(t, x_0) = \sqrt{x_0^2 - 2t}$$

We left the domain of definition of $-1/x$.

Example 3

$\dot{x} = -\sqrt{x}$ $\qquad J = \mathbb{R}$, $\Omega = \{x \geq 0\}$



$\phi(0, x_0) = x_0$

$\phi(t, 0) = 0$ is a soln.

$\phi(t, x_0) \neq 0$

$$\frac{d}{dt}\phi(t, x_0) = -\sqrt{\phi(t, x_0)} \Rightarrow \frac{\frac{d}{dt}\phi(t, x_0)}{\sqrt{\phi(t, x_0)}} = -1$$

$$\Rightarrow \frac{d}{dt}\left(2\sqrt{\phi(t, x_0)}\right) = -1$$

$$\int_0^t \frac{d}{ds}\left(2\sqrt{\phi(s, x_0)}\right) ds = -t \Rightarrow \sqrt{\phi(t, x_0)} - \sqrt{x_0} = -t/2$$

$$\phi(t, x_0) = \left(\sqrt{x_0} - t/2\right)^2$$

$\boxed{\begin{aligned} &\dot{x} = -\sqrt{x} \\ &x(t_0) = 0 \end{aligned}}$ The soln is not unique.

There are two solutions going through the same point.

HW Expain how to construct an infinite amount of solns satisfying this IVP.

Contraction

Existence and uniqueness theorem

($\exists$, !, smooth dependence on initial conditions & parameters)

- Banach fixed point theorem.
- Grönwall's inequality.

Let X be a complete metric space with distance $d(x_1, x_2) = |x_1 - x_2|$

Def

Let $T: X \to X$, then T is a contraction if there exists a $\lambda \in (0,1) \in \mathbb{R}$ such that

$$d(T(x), T(y)) \leq \lambda d(x,y), \quad \forall x,y \in X$$

$\to$ The contracting mapping theorem states that any contraction from a complete metric space back to itself has a unique fixed point. $\exists\, x_0 \in X$ s.t. $T(x_0) = x_0$ and its unique.

Thm 1.2

If the function T is a contraction of the metric space (X, d), with contracting constant λ, then there is a unique $x_0 \in X$ s.t. $T(x_0) = x_0$.

Moreover, the sequence $\{T^n(x)\}$ (sequence) convergece to x_0 when $n \to \infty$ with a rate given by $d(T^n(x), x_0) \leq \frac{\lambda^n}{1-\lambda} d(T(x), x)$

Banach fixed point theorem

Proof (Banach fixed point theorem)

i) First, the point is unique.

Let x_0, x_1 be fixed points $T(x_0)=x_0,\ T(x_1)=x_1$

$$d(x_0, x_1) = d(T(x_0), T(x_1)) \le \lambda\, d(x_0, x_1) < d(x_0, x_1)$$

That's a contradiction, therefore there is only one.

ii) The point exists. Build a Cauchy sequence from $\{T^n(x)\}$.

$T^{n+1} = T \circ T^n$

$$d(T^{n+1}(x), T^n(x)) \le \lambda\, d(T^n(x), T^{n-1}(x)) \le \lambda^2 d(T^{n-1}(x), T^{n-2}(x)) \le \dots \le \lambda^n d(T(x), x)$$

$$d(T^{n+p}(x), T^n(x)) \overset{t.i.}{\le} d(T^{n+p}(x), T^{n+p-1}(x)) + d(T^{n+p-1}(x), T^{n+p-2}(x)) + \dots + d(T^{n+1}(x), T^n(x)) \le (\lambda^{n+p-1} + \lambda^{n+p-2} + \dots + \lambda^n)\, d(T(x), x)$$

$$= \lambda^n (\lambda^{p-1} + \lambda^{p-2} + \dots + \lambda + 1)\, d(T(x), x) < \lambda^n \left(\sum_{k=0}^{\infty} \lambda^k\right) d(T(x), x)$$

$$= \lambda^n \left(\frac{1}{1-\lambda}\right) d(T(x), x) < \varepsilon$$ then $\{T^n(x)\}$ is Cauchy

$$\Rightarrow T^n(x) \to x_0$$

If T^n was a dynamical system then x_0 would be a global attractor.

x_0 is a fixed point

$$\lim_{n\to\infty} T^{n+1}(x) = \lim_{n\to\infty} T^n(x) = x_0$$

We also know that

$$d(T^{n+1}(x), T(x_0)) \le \lambda\, d(T^n(x), x_0) \xrightarrow[n\to\infty]{} 0$$

$$\lim_{n\to\infty} T^n(x) = x_0 \qquad \lim_{n\to\infty} T^{n+1}(x) = T(x_0)$$

and the limit is unique so $T(x_0) = x_0$

Finally the rate of convergence is given by

$$d(T^{n+p}(x), T^n(x)) \le \lambda^n \left(\frac{1}{1-\lambda}\right) d(T(x), x)$$

$$\Big\downarrow\ p \to \infty$$

$$d(x_0, T^n(x)) \le \lambda^n \left(\frac{1}{1-\lambda}\right) d(T(x), x)$$

Gronwall's inequality

Thm (Grönwall's inequality)

Let α, ϕ, ψ be continuous non-negative functions of the interval $[a,b]$ (non-empty $a<b$).

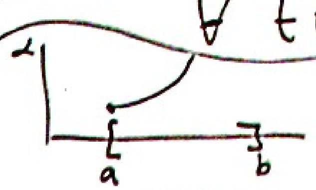
Suppose that α is continuously differentiable and non-decreasing in the interior of (a,b).

If for every $t \in [a,b]$,

$$\phi(t) \le \alpha(t) + \int_a^t \psi(s)\,\phi(s)\,ds$$

Then

$$\phi(t) \le \alpha(t)\, e^{\int_0^t \psi(s)\,ds} \quad \forall\, t \in [a,b]$$



Pf Suppose that $\alpha(a) > 0$

which implies that $\alpha(t) \ge \alpha(a) > 0, \ \forall t \in [a,b]$

$$\alpha(t) + \int_a^t \psi(s)\phi(s)\,ds > 0$$

$$\Rightarrow \frac{\phi(t)\,\psi(t)}{\psi(t)\left(\alpha(t) + \int_a^t \psi(s)\phi(s)\,ds\right)} \le 1 \quad \text{then if } \psi(t) \ne 0$$

$$\Rightarrow \frac{\phi(t)\psi(t) + \dot\alpha(t) - \dot\alpha(t)}{\psi(t)\left(\alpha(t) + \int_a^t \psi(s)\phi(s)\,ds\right)} \le 1 \quad \left(\begin{array}{l}\text{This is also true} \\ \text{when } \psi(t)=0\end{array}\right)$$

$$\Rightarrow \frac{\phi(t)\psi(t) + \dot\alpha(t)}{\alpha(t) + \int_a^t \psi(s)\phi(s)\,ds} \le \psi(t) + \frac{\dot\alpha(t)}{\alpha(t) + \int_a^t \psi(s)\phi(s)\,ds} \le \psi(t) + \frac{\dot\alpha(t)}{\alpha(t)}$$

$$\Rightarrow \frac{d}{dt}\log\left(\alpha(t) + \int_a^t \psi(s)\phi(s)\,ds\right) \le \psi(t) + \frac{d}{dt}\log(\alpha(t))$$

Integrating from $[a,t]$

$$-\log(\alpha(a)) + \log\left(\alpha(t) + \int_a^t \psi(s)\phi(s)\,ds\right) \le \int_a^t \psi(s)\,ds - \log(\alpha(a)) + \log(\alpha(t))$$

$$\log\left(\alpha(t) + \int_a^t \psi(s)\phi(s)\,ds\right) - \log(\alpha(t)) \le \int_a^t \psi(s)\,ds$$

$$\overset{\text{algebra}}{\Rightarrow} \phi(t) \le \alpha(t) + \int_a^t \psi(s)\phi(s)\,ds \le \alpha(t)\, e^{\int_a^t \psi(s)\,ds}$$

Grönwall

To prove that this is true for $\alpha(a) = 0$ we asume $\alpha(a) = \varepsilon \to 0$ the limit is still true