Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

Renato Calleja, 6 de marzo de 2024

$\dot{x} = A(t)x + g(x,t)$, $\quad A(t+T) = A(t) \ \forall t$

$g(x, t+T) = g(x,t)$

$g(x,\cdot) = O(x^2)$

Floquet Theory and linearization around a Periodic orbit.

Suppose that $\dot{x} = f(x,t)$, $x \in \mathbb{R}^n$

- smooth (at least C^1)
- Periodic in t. $\exists\, T>0$ s.t.

$f(x, t+T) = f(x,t)$, $\forall x \in \mathbb{R}^n, t \in \mathbb{R}$ (non-autonomous but periodic w.r.t. t)

The system can be made autonomous.

$$\begin{aligned} \dot{x} &= f(x,\psi) \\ \dot{\psi} &= 1 \quad (\mathrm{mod}\ T) \end{aligned} \quad (1) \qquad \psi \in [0,T] \text{ with the endpoints identified, homeomorphic to the circle } S \text{ (or } \mathbb{T})$$

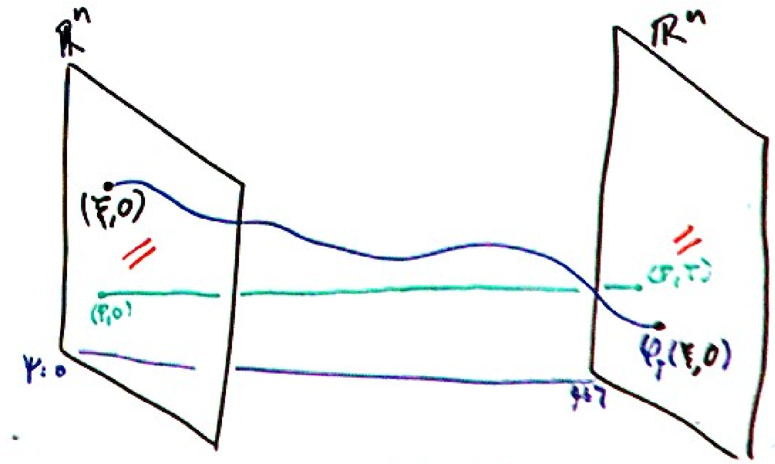
ψ is an angular variable modulus T.

The phase space is a phase cylinder $\mathbb{R}^n \times S$ (or $\mathbb{R}^n \times \mathbb{T}$)

Notice that $\psi + nT = \psi$, $\forall n \in \mathbb{Z}$

$\mathbb{T}$ is $\mathbb{R}$ with points identified in this way

$\mathbb{T} = \mathbb{R} \bmod (T\mathbb{Z})$ $\qquad$ φ is the flow of (1)



Poincaré map $\quad \mathcal{P} : \Sigma \to \Sigma$

$\xi \in \mathbb{R}^n \qquad \mathcal{P}(\xi) = \varphi_T(\xi, 0)$ $\left(\text{in fact this is equal to } \Pi_x \varphi_T(\xi,0) \text{ projection.}\right)$

The set $\Sigma = \{(\xi,\psi) \mid \psi = 0\}$

What happens if $\exists\, p \in \mathbb{R}^n$ $\quad f(p,t) = 0$, $\forall t \in \mathbb{T}$

$\mathcal{P}(p) = p$

$$\dot X = A(t)X + g(X,t), \quad A(t+T)=A(t) \ \forall t$$
$$g(X,t+T)=g(X,t)$$
$$g(x,\cdot)=O(x^2)$$

Floquet Theory and linearization around a Periodic orbit.

The derivative of $\mathcal{P}$ w.r.t. ξ is a linear transformation in $\mathbb{R}^n$ given by $D_\xi \mathcal{P}(p) = D_\xi \varphi_T(p,0)$
↳ nxn matrix

Poincaré: "It is easy to see that $D_\xi \varphi_t(p,t)$ is the principal fundamental solution matrix of the problem

The first variation of $\dot x = f(x,t)$
$$\begin{cases} \dot W = \underbrace{D_\xi f(\varphi_t(p,t))}_{\text{this is a T-periodic matrix.}} W \\ W(0)=I \end{cases}$$
"

Then $D_\xi \varphi_t(p,t) = P(t)e^{tB}$

if $t=T$ $\quad D_\xi \varphi_T(p,0) = e^{TB}$

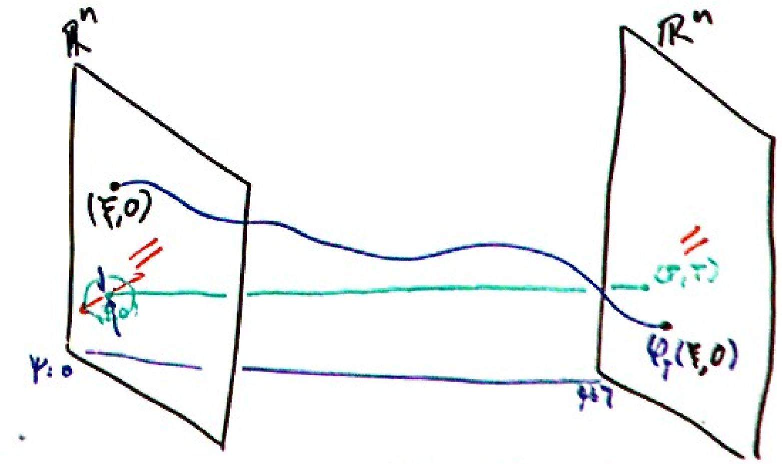
The Floquet multipliers coincide with the eigenvalues of the Poincaré map.

Notice that $\psi + nT = \psi$, $\forall n \in \mathbb{Z}$ ← in this way

$\mathbb{T}$ is $\mathbb{R}$ with points identified

$\mathbb{T} = \mathbb{R} \bmod (T\mathbb{Z})$

φ is the flow of (1)



Poincaré map $\mathcal{P}: \Sigma \to \Sigma$ $\left(\text{in fact this is equal to } \Pi_x \varphi_T(\xi,0)\right)$
$\xi \in \mathbb{R}^n$ $\quad \mathcal{P}(\xi) = \varphi_T(\xi,0)$ $\quad$ ↳ projection.

The set $\Sigma = \{(\xi,\psi) \mid \psi = 0\} \simeq \mathbb{R}^n$

What happens if $\exists p \in \mathbb{R}^n$ $\quad f(p,t)=0, \ \forall t \in \mathbb{T}$

$\mathcal{P}(p) = p$

Lyapunov exponents

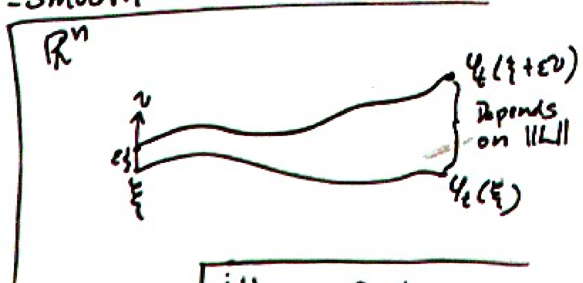
(Aleksandr Mikhailovich Lyapunov)

Generalization of Floquet exponents to solutions that are not necessarily periodic.

Let the nonlinear differential equation

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad f\text{-smooth} \qquad (2)$$

with flow φ_t.



If $\varepsilon \in \mathbb{R}$, $\xi, v \in \mathbb{R}^n$ and $\eta := \xi + \varepsilon v$

the two solutions start at nearby points.

$$t \mapsto \varphi_t(\xi), \quad t \mapsto \varphi_t(\underbrace{\xi + \varepsilon v}_{\eta})$$

¿How fast are these two solutions drifting apart?

By Taylor's theorem (around $\varepsilon = 0$) we have,

$$\varphi_t(\xi + \varepsilon v) - \varphi_t(\xi) = \varepsilon D_\xi \varphi_t(\xi) v + O(\varepsilon^2)$$

(derivative of $x \mapsto \varphi_t(x)$)

Following Poincaré, $t \mapsto D_\xi \varphi_t(\xi)$ is the principal fundamental solution matrix of

$$\dot{W} = D f(\varphi_t(\xi)) W, \quad W(0) = I_d \qquad (3)$$

Excersise Check that if φ is the flow of (2) then $D_\xi \varphi_t(\xi)$ is p.f.s.m. of (3).

We define the linear operator L for $v \neq 0$, $a \in \mathbb{R}$, $v \in \mathbb{R}^n$

$$L(av) = D\varphi_t(\xi) av$$

The operator norm measures the "expansion" or "contraction" of a vector,

$$\|L\| = \sup_{a \neq 0} \frac{|D\varphi_t(\xi) av|}{|av|} = \frac{|D\varphi_t(\xi) v|}{|v|}$$

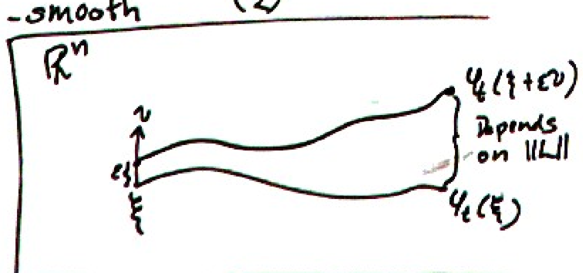
Lyapunov exponents

(Aleksandr Mikhailovich Lyapunov)

Generalization of Floquet exponents to solutions that are not necessarily periodic.

Let the nonlinear differential equation

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad f\text{-smooth} \quad (2)$$



$$\varphi_t(\xi) = \xi + \int_0^t f(\varphi_s(\xi))ds$$

$$\varphi_t(\xi+\varepsilon v) = \xi + \varepsilon v + \int_0^t f(\varphi_s(\xi+\varepsilon v))ds$$

$$|\varphi_t(\xi+\varepsilon v) - \varphi_t(\xi)| \leq \varepsilon|v| + \int_0^t |f(\varphi_s(\xi+\varepsilon v)) - f(\varphi_s(\xi))|ds$$

$$\leq \varepsilon|v| + Lip(f)\int_0^t |\varphi_s(\xi+\varepsilon v) - \varphi_s(\xi)|ds$$

$$|\varphi_t(\xi+\varepsilon v) - \varphi_t(\xi)| \leq \varepsilon|v| \exp(t\, Lip(f))$$

¿How fast are these two solutions drifting apart?

By Taylor's theorem (around $\varepsilon=0$) we have,

$$\varphi_t(\xi+\varepsilon v) - \varphi_t(\xi) = \varepsilon D_\xi \varphi_t(\xi)v + O(\varepsilon^2)$$

Definition

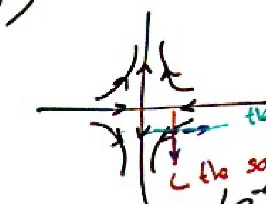
Let $\xi \in \mathbb{R}^n$ and a soln to the diff. eq. $\dot{x} = f(x) \dots (2)$ defined for every $t \geq 0$. Also let $v \in \mathbb{R}^n$ be a non-zero vector. The Lyapunov exponent in the direction of v for φ is defined by,

$$\chi(p, v) := \limsup_{t\to\infty} \frac{1}{t} \log\left(\frac{|D\varphi_t(\xi)v|}{|v|}\right)$$

$$\|L\| = \sup_{a\neq 0} \frac{|D\varphi_t(\xi)av|}{|av|} = \frac{|D\varphi_t(\xi)v|}{|v|}$$

Lyapunov exponents

(Aleksandr Mikhailovich Lyapunov)



Example

$\dot{x} = -ax$, $\dot{y} = by$, $a, b > 0$

$\varphi_t(x,y) = (e^{-at}x, e^{bt}y)$ $D\varphi_t(x,y) = \begin{pmatrix} e^{-at} & 0 \\ 0 & e^{bt} \end{pmatrix}$

$v = (w, z)$

$w = 0$, $z \neq 0$ $\chi(P,v) = \limsup_{t\to\infty} \frac{1}{t} \log\left(\frac{\left| \begin{pmatrix} e^{-at} & 0 \\ 0 & e^{bt} \end{pmatrix} \begin{pmatrix} 0 \\ z \end{pmatrix} \right|}{|z|} \right) = b$

$w \neq 0$, $z = 0$ $\boxed{\chi(P,v) = -a}$

Tomorrow we do it for Floquet exponents.

By Taylor's theorem (around $\varepsilon = 0$) we have,

$$\varphi_t(\xi + \varepsilon v) - \varphi_t(\xi) = \varepsilon D_\xi \varphi_t(\xi) v + O(\varepsilon^2)$$

Definition

Let $\xi \in \mathbb{R}^n$ and a soln to the diff. eq. $\dot{x} = f(x)$ ... (2) defined for every $t \geq 0$. Also let $v \in \mathbb{R}^n$ be a non-zero vector. The Lyapunov exponent in the direction of v for φ is defined by,

$$\chi(P,v) := \limsup_{t\to\infty} \frac{1}{t} \log\left(\frac{|D\varphi_t(\xi) v|}{|v|} \right)$$

$$\|L\| = \sup_{a \neq 0} \frac{|D\varphi_t(\xi) a v|}{|a v|} = \frac{|D\varphi_t(\xi) v|}{|v|}$$