Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

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Stable, unstable and center manifolds

Let $\dot{x} = f(x)$ with $\varphi : \mathbb{R} \times \mathbb{R}^L \to \mathbb{R}^L$

$\tilde{x}$ equilibrium

The stable and unstable spaces are given by

$$W^s(\tilde{x}) = \{ x \in \mathbb{R}^L \mid \lim_{t \to \infty} \varphi(t,x) = \tilde{x} \}$$

$$W^u(\tilde{x}) = \{ x \in \mathbb{R}^L \mid \lim_{t \to -\infty} \varphi(t,x) = \tilde{x} \}$$

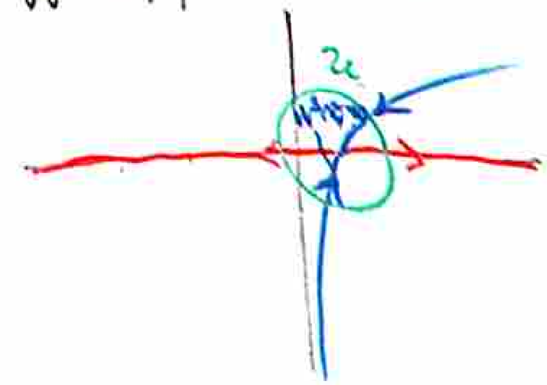
How do we characterize this spaces?
Locally these spaces have nicer properties

Def

$\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ with fixed point $\tilde{x}$.
Given a neighborhood $\mathcal{U}$ of $\tilde{x}$ the associated local stable and unstable manifolds are,

$$W^s_{loc}(\tilde{x}) = W^s(\tilde{x}, \mathcal{U}) = \{ x \in W^s(\tilde{x}) \mid \varphi([0,\infty), x) \subset \mathcal{U} \}$$

$$W^u_{loc}(\tilde{x}) = W^s(\tilde{x}, \mathcal{U}) = \{ x \in W^u(\tilde{x}) \mid \varphi((-\infty, 0], x) \subset \mathcal{U} \}$$



Stable, unstable and center manifolds

Thm ((Un)stable manifold)
Let $\dot{x} = f(x)$, $x \in \mathbb{R}^L$, $f \in C^k(\mathbb{R}^L)$, $k = \{1, 2, \dots, \infty\} \cup \{\omega\}$
for which $f(0) = 0$. Assume that

$$Df(0) = \begin{pmatrix} A^u & 0 \\ 0 & A^s \end{pmatrix}$$

is hyperbolic with
$A^u \in \mathcal{L}(\mathbb{R}^d)$ eigvals with pos real part
$A^s \in \mathcal{L}(\mathbb{R}^{L-d})$ eigvals with neg real part.

Then there exists a neighborhood $\mathcal{U} = \mathcal{U}^s \times \mathcal{U}^u \subset \mathbb{R}^d \times \mathbb{R}^{L-d}$ of the origin and C^k functions $P^u : \mathcal{U}^u \to \mathbb{R}^L$, $P^s : \mathcal{U}^s \to \mathbb{R}^L$ that are tangent to $\mathbb{R}^d \times \{0\}$ and $\{0\} \times \mathbb{R}^{L-d}$ at 0 respectively and such that

$$W^u_{loc}(0) = P^u(\mathcal{U}^u)$$
$$W^s_{loc}(0) = P^s(\mathcal{U}^s)$$

There are 2 main ways to prove this theorem.

1) Lyapunov–Perron method (Chicone 06)
2) Hadamard graph transform.

There is also the parameterization method
Cabré, Fontich, de la Llave 2001

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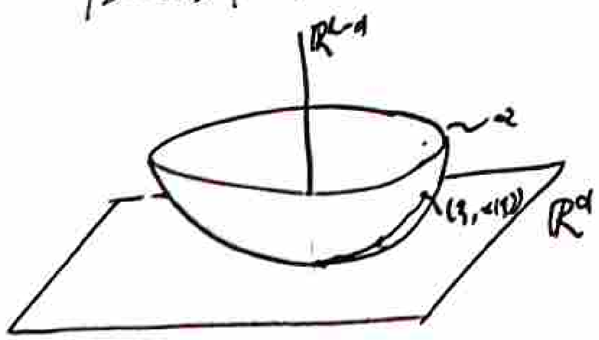
Another consequence of the theorm

Moreover, if $(\xi, \alpha(\xi)) \in W(0,0)$ then $\forall \lambda > a$ $\exists$ a constant $C>0$ s.t. the solution $t \to (x(t), y(t))$ of $(*)$ with initial condition $(\xi, \alpha(\xi))$ satisfies that

$$|x(t)| + |y(t)| \leq C e^{\lambda t} |\xi|$$



Def $\|F\|_1 = \sup_{x \in \mathbb{R}^n} (|F(x)| + |DF(x)|)$

Thm [Chicone 06, Thm 4.1]

Suppose that a & b real numbers s.t. $a<0<b$, $S: \mathbb{R}^d \to \mathbb{R}^d$ & $U: \mathbb{R}^{L-d} \to \mathbb{R}^{L-d}$ real transformations.

S has eigvals with real part $<a$.

U has eigenvals with real part $>b$.

If $F \in C^1(\mathbb{R}^d \times \mathbb{R}^{L-d}, \mathbb{R}^d)$ & $G \in C^1(\mathbb{R}^d \times \mathbb{R}^{L-d}, \mathbb{R}^{L-d})$

$$\begin{cases} \dot{x} = Sx + F(x,y) \\ \dot{y} = Uy + G(x,y) \end{cases} \quad (*)$$

$F(0,0) = 0$, $DF(0,0) = 0$, $G(0,0) = 0$, $DG(0,0) = 0$

$\|F\|_1$ & $\|G\|_1$ are sufficiently small, then there is a unique function $\alpha \in C^1(\mathbb{R}^d, \mathbb{R}^{L-d})$ s.t. $\alpha(0) = 0$, $D\alpha(0) = 0$, $\sup_{\xi \in \mathbb{R}^d} |D\alpha(\xi)| < \infty$

& the graph of α is given by

$$W(0,0) = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^{L-d} : y = \alpha(x)\}$$

W is an invariant manifold for the system of equations $(*)$.

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Let $\dot{\tilde{y}} = f(\tilde{y})$, $\tilde{y} \in \mathbb{R}^L$, $f \in C^2$

p is an equilibrium point $f(p)=0$ that is hyperbolic.

We define $\tilde{y} = p + z$

$$\dot{\tilde{y}} = \dot{z} = Df(p)z + \mathcal{N}(z)$$

$$\mathcal{N}(z) = f(p+z) - Df(p)z$$

Since p is hyperbolic $\exists$ a B such that (Jordan Canonical Form)

$$B\, Df(p)\, B^{-1} = \begin{pmatrix} S & 0 \\ 0 & U \end{pmatrix}$$

Now we impose $w = Bz \Rightarrow B^{-1}w = z$

$$\dot{w} = B\dot{z} = B\, Df(p) B^{-1} w + B\mathcal{N}(B^{-1}w) = \begin{pmatrix} S & 0 \\ 0 & U \end{pmatrix} w + B\mathcal{N}(B^{-1}w)$$

$$= \begin{pmatrix} S & 0 \\ 0 & U \end{pmatrix} w + \tilde{\mathcal{N}}(w)$$

$$\tilde{\mathcal{N}}(w) = B\mathcal{N}(B^{-1}w)$$

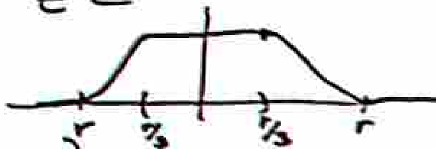
$x = (w_1, \dots, w_d) \in \mathbb{R}^d$ and $y = (w_{d+1}, \dots, w_L) \in \mathbb{R}^{L-d}$

$$\tilde{F}(x,y) = (\tilde{\mathcal{N}}_1(x,y), \dots, \tilde{\mathcal{N}}_d(x,y)) \in \mathbb{R}^d$$

$$\tilde{G}(x,y) = (\tilde{\mathcal{N}}_{d+1}(x,y), \dots, \tilde{\mathcal{N}}_L(x,y)) \in \mathbb{R}^{L-d}$$

$$\dot{x} = Sx + \tilde{F}(x,y) \qquad \tilde{F}, \tilde{G} \in C^1$$

$$\dot{y} = Ux + \tilde{G}(x,y)$$



Let $\gamma : \mathbb{R}^d \times \mathbb{R}^{L-d} \to \mathbb{R}$, $\gamma \in C^1$

$$\gamma(x,y) = \begin{cases} 1 & \text{if } (x,y) \in B_{r/3} \\ 0 & \text{if } (x,y) \notin B_r \end{cases}$$

Bump function. (Función flan)

$\|\gamma\|_0 = 1$, $\|D\gamma\|_0 \leq 2/r$

$F(x,y) = \gamma(x,y)\tilde{F}(x,y)$

$G(x,y) = \gamma(x,y)\tilde{G}(x,y)$

$$\dot{x} = Sx + F(x,y)$$
$$\dot{y} = Uy + G(x,y)$$

$F, G \in C^1$, $\|F\|_1, \|G\|_1$ as small as needed

$F(0,0) = 0 = G(0,0)$ $DF(0,0) = 0 = DG(0,0)$

$\alpha(0) = 0$; $D\alpha(0) = 0$, $\sup_{\xi \in \mathbb{R}^d} |D\alpha(\xi)| < \infty$

& the graph of α is given by

$$W(0,0) = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^{L-d} : y = \alpha(x)\}$$

W is an invariant manifold for the system of equations (*).

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Example

$\dot{x} = 2x - (2+y)e^{y} = f\binom{x}{y}$

$\dot{y} = -y$

We know that there exists a matrix B

$$B\,Df\binom{1}{0}B^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \quad (JCF)$$

$$\binom{x}{y} = \binom{1}{0} + \binom{z_1}{z_2} \qquad \binom{W^s}{W^u} = B\binom{z_1}{z_2}$$

Volunteer
Finish all the computation of this example.

$\dot{W}_s = -W_s$

$\dot{W}_u = 2W_u + (2 + 3W_s - (2+W_s)e^{W_s})$

The function α is given by, $W_u = \alpha(W_s)$

$\Rightarrow \frac{d}{dt}\alpha(W_s) = 2\alpha(W_s) + (2+3W_s - (2+W_s)e^{W_s})$

The stable manifold $x = e^{y} \rightsquigarrow W_u = x-y-1,\ W_s = y$

$\alpha(W_s) = e^{W_s} - W_s - 1 \rightarrow$ It satisfies the diff eq..

Stable, unstable and center manifolds

Proof

We want α such that $y=\alpha(X) \Rightarrow \dot{X} = SX + F(X,y)$, $\dot{y} = Uy + G(X,y)$

Then we must have that $\dot{X} = SX + F(X,\alpha(X))$

Given α let $\mathcal{X}(t,\xi;\alpha)$ be the solution to the IVP with initial condition ξ.

α also satisfies $\frac{d}{dt}[\alpha(\mathcal{X})] = U\alpha(\mathcal{X}) + G(\mathcal{X},\alpha(\mathcal{X}))$

$$e^{-Ut}\frac{d}{dt}[\alpha(\mathcal{X})] - e^{-Ut}U\alpha(\mathcal{X}) = e^{-Ut}G(\mathcal{X},\alpha(\mathcal{X}))$$

$$\Rightarrow \int_0^t \frac{d}{ds}\left[e^{-Us}\alpha(\mathcal{X})\right]ds = \int_0^t e^{-Us}G(\mathcal{X},\alpha(\mathcal{X}))\,ds$$

$$\Rightarrow e^{-Ut}\alpha(\mathcal{X}(t)) - \alpha(\xi) = \int_0^t e^{-Us}G(\mathcal{X},\alpha(\mathcal{X}))\,ds$$

$$\xrightarrow{t\to\infty} \quad \alpha(\xi) = -\int_0^\infty e^{-Us}G(\mathcal{X},\alpha(\mathcal{X}))\,ds \quad \text{(Lyapunov – Perron operator)}$$

How to formalize all this.

1st) in order to have a solution $\mathcal{X}$, $F(X,\alpha(X))$ should be Lipschitz w.r.t. X

→ α should belong to a space of Lipschitz functi

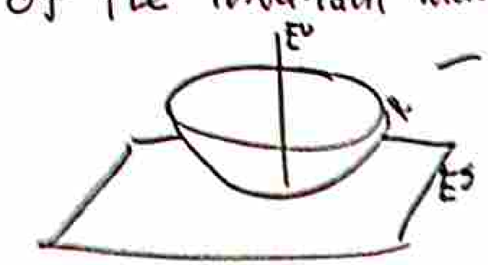
2nd) The semigroup e^{-Ut} goes to zero as $t\to\infty$

$\|e^{-Ut}\| \le Ce^{-\lambda t}$. use these estimates.

3rd) Prove that the L-P operator has a fixed point.

→ We'll work in Banach spaces of Continuous & Lipschitz functions.

4th) Prove that the fixed point is the graph of the invariant manifold.



— invariant tangent !!