Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

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Non-linear global theory

Goal: Say something about the asymptotic behaviour of a system.

1) Linear systems. We can do it!

2) We can solve the question in a neighborhood of a simple solution.

There are at least 2 cases where we have a descrition of a global behaviour.

- Gradient systems (in any dimension)
- Hamiltonian systems (in 2 dimensions)

Hamiltonian Systems

Consider the function

$$H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$(q_1, \dots, q_n, p_1, \dots, p_n) \mapsto H(q_1, \dots, q_n, p_1, \dots, p_n),$$

and the system of diff. eqns.

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad i = 1, \dots, n$$

(System of 2n-diff. equ

This system is called a Hamiltonian system (of n degrees of freedom).

Non-linear global theory

Notice that the system can be written in the following way,

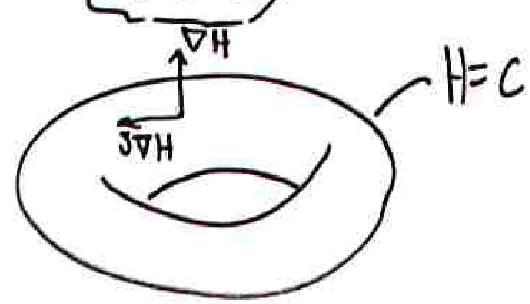
$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = J \nabla H(q,p), \quad J = \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix}$$

J is called a symplectic matrix.

$$J^2 = -Id_{2n}, \quad J J^T = Id_{2n}$$

$\{ i(-i) = 1 \}$

$$J^T = -J$$



Hamiltonian Systems

$H(q,p)$ is the total energy of the system

Thm
The total energy $H(q,p)$ remains constant along trajectories of the Hamiltonian system. $\left(\dot{q}_j = \frac{\partial H}{\partial p_j}, \dot{p}_j = -\frac{\partial H}{\partial q_j}\right)$

Pf

$$\frac{d}{dt} H(q(t), p(t)) = \sum_{j=1}^{n} \frac{\partial H}{\partial q_j} \cdot \dot{q}_j + \frac{\partial H}{\partial p_j} \cdot \dot{p}_j$$

$$= \sum_{j=1}^{n} \frac{\partial H}{\partial q_j} \cdot \frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial p_j} \cdot \frac{\partial H}{\partial q_j} = 0$$

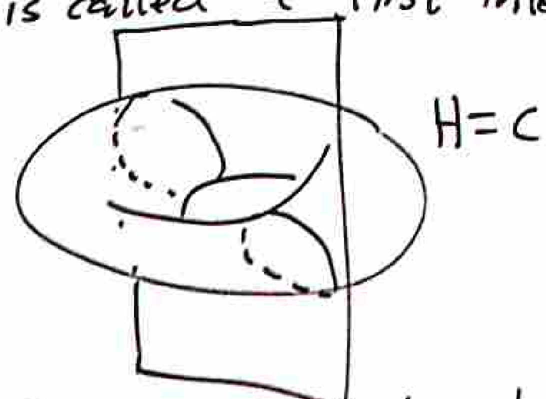
$$\Rightarrow H(q(t), p(t)) = C.$$

$$\frac{d}{dt} H(q,p) = \nabla H(q,p) \cdot \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \nabla H(q,p) \cdot J \nabla H(q,p) = 0$$

Non-linear global theory

In general, if $F: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ has a gradient that is orthogonal to the Hamiltonian vector field, then the level sets of F are also invariant.

Then F is called a "first integral" of the system.



If we can find enough first integrals, then we can write the solution completely.

If we have a collection of n first integrals and they are all lin. indep. at every point then they impose restrictions to the solution trajectories and we can thereafter write down a formula for the solution to the Ham. Sys.

$\Rightarrow$ Integrability (Jacobi)

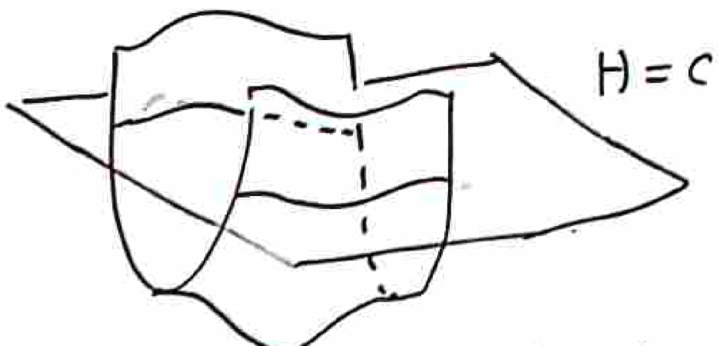
Non-linear global theory

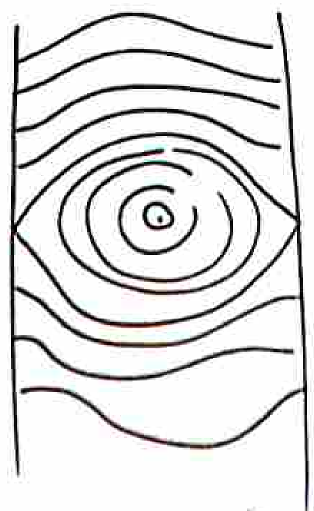
Example (In 2 dimensions the Hamiltonian is enough)

$\dot{q} = p$

$\dot{p} = -\sin(q)$ pendulum

$$H(q,p) = \frac{p^2}{2} - \cos(q)$$





H is a first integral, but single system is 2 dim (1-dof) then the solution is completely determined.

Fixed points of Hamiltonian systems

All the fixed points of Hamiltonian systems are either centers or saddles

In order to see this we first notice that the fixed points of a Ham. v.f. are zeros of ∇H.

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = J \nabla H(q,p)$$

Let's compute the Jacobian of $J\nabla H$.

$$D(J\nabla H) = J D^2 H$$

$$D^2 H = \begin{pmatrix} \frac{\partial^2 H}{\partial q_1^2} & \cdots & \frac{\partial^2 H}{\partial p_n \partial q_1} \\ \vdots & & \vdots \\ \frac{\partial^2 H}{\partial p_1 \partial q_n} & & \vdots \\ \vdots & & \\ \frac{\partial^2 H}{\partial q_1 \partial p_n} & \cdots & \frac{\partial^2 H}{\partial p_n^2} \end{pmatrix}$$

→ Hessian of H.

If H is smooth enough (C^2) then $D^2 H = {D^2 H}^T$.

Non-linear global theory

If $A = J D^2 H(q^*, p^*)$, (q^*, p^*) fixed point.

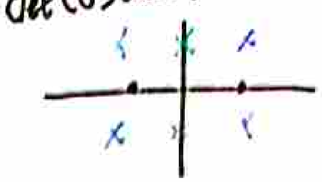
$$A^T J + JA = D^2H^T J^T J + J J D^2 H = D^2 H^T - D^2 H = 0$$

$$A^T J + JA = 0 \iff J A^T J - A = 0$$

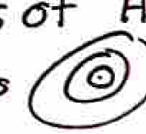
$$\iff J A^T J = A, \quad A^T = D^2 H^T J^T$$

<u>Proposition</u>
The characteristic polynomial of a fixed point of a Hamiltonian system, $P(\lambda)$, is an even function ($P(\lambda) = P(-\lambda)$), then if λ is a root of P ($P(\lambda)=0$) then so is $-\lambda$, $\bar{\lambda}$, $-\bar{\lambda}$.

<u>Pf</u> $P(\lambda) = \det(A - \lambda I) = \det(J A^T J + \lambda J J) = \det(J)\det(A^T + \lambda I)\det(J)$
$= \det(A^T - (-\lambda) I) = P(-\lambda)$ //



Fixed points of Hamiltonian systems

All the fixed points of Hamiltonian systems are either centers  or saddles 

In order to see this we first notice that the fixed points of a Ham. v.f. are zeros of ∇H.

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If H is smooth enough (C^2) then $D^2 H = D^2 H^T$.

Non-linear global theory

Physical systems of masses satisfying

$$M\ddot{x} + \nabla V(x) = g(t)$$ └ forcing term

These are all Hamiltonian systems.

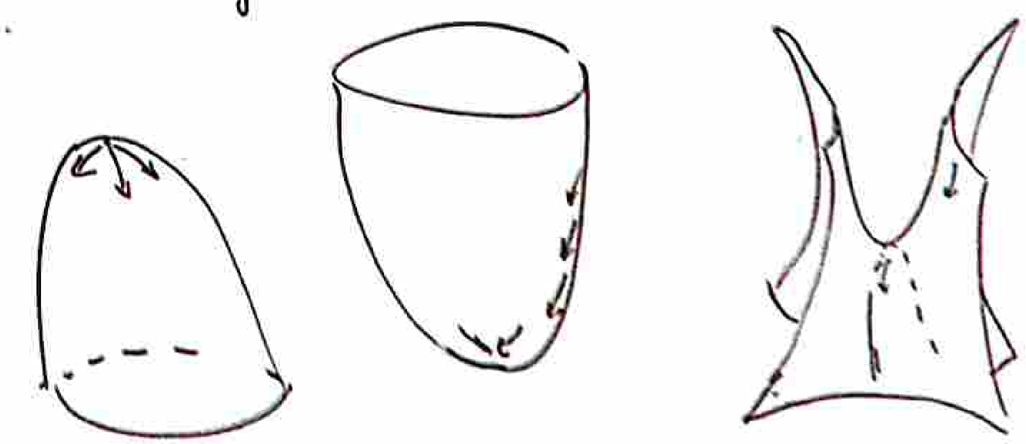
$$M = \begin{pmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_n \end{pmatrix}, \qquad \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = J\nabla H(q,p)$$

Gradient flows and Lyapunov functions.

Let $G: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth function (n is not necessarily even). Then an associated gradient system is,

$$\dot{x} = -\nabla G(x)$$

- Sometimes we can omit the minus sign.
- With the minus, the v.f. is pointing towards the greatest descent direction.

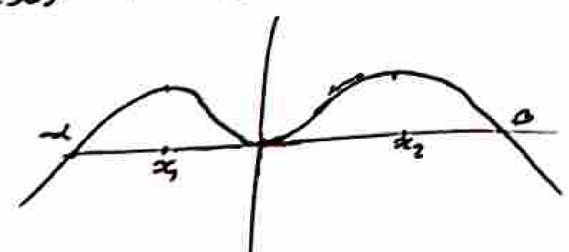


Non-linear global theory

<u>Example</u>

Consider the gradient flow

$$G(x) = x^2 (x+\alpha)(x-\beta), \quad \alpha, \beta > 0$$



$$\dot{x} = -G'(x) = -4x(x-x_1)(x-x_2)$$

The motion of a gradient flow is like the motion of a ball inside some very viscous liquid.

Gradient flows and Lyapunov functions.

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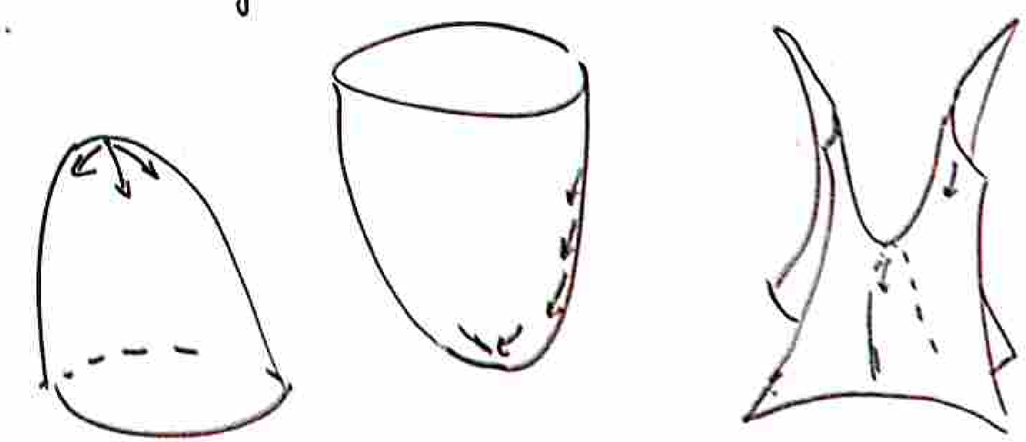

Non-linear global theory

The fixed points of a gradient flow satisfy that $DG(x^*) = 0$, and the linearization around these fixed point is related to $D^2G(x^*)$.

Notice again that $B = D^2G(x^*)$, $B^T = B$.

Lemma

All the eigenvalues of B are real.

Pf

Consider the inner product of $\mathbb{C}^n$. $\langle \cdot, \cdot \rangle$

Now take $v, w \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$.

i) $\langle \lambda v, w \rangle = \lambda \langle v, w \rangle$

ii) $\langle v, \lambda w \rangle = \bar{\lambda} \langle v, w \rangle$

iii) $\langle v, w \rangle = \overline{\langle w, v \rangle}$

Consider $\lambda \in \mathbb{C}$, s.t. $Bv = \lambda v$ (v eigenvector, λ eigenvalue)

$$\lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle Bv, v \rangle = \langle v, Bv \rangle = \langle v, \lambda v \rangle = \bar{\lambda} \langle v, v \rangle$$

$\lambda = \bar{\lambda}$ so λ has to be real. //

The only possibilities are stable or unstable nodes or saddle. (No centers for G.S.).