# Ecuaciones Diferenciales Ordinarias

## Posgrado en ciencias matemáticas UNAM

## IIMAS

**Renato Calleja, 28 de febrero de 2024**

<u>Thm</u> (Floquet's Theorem)

$\Phi(t)$ fundamental matrix solution of

$\dot{x} = A(t)x$, $x \in \mathbb{R}^n$, $A(t+T) = A(t)$

then $\forall t \in \mathbb{R}$

$$\Phi(t+T) = \Phi(t)\underbrace{\Phi^{-1}(0)\,\Phi(T)}_{e^{TB}}$$

$e^{TB}$ ← possibly complex

$$\Phi(t) = P(t)\, e^{Bt} \quad \forall t$$

• $R$ real matrix and a $2T$-periodic $Q(t)$ real.

$$\Phi(t) = Q(t)\, e^{Rt}.$$

We will prove the thm using logs of matrices.

<u>Prop</u> (2.82 - Chicone '06)

If $C$ is a non-singular $n \times n$ matrix, then there is an $n \times n$ possibly complex matrix $B$ such that $e^B = C$. ($B = \log C$)

In addition if $C$ is real, then there is a real matrix $B$ such that $e^B = C^2$.

<u>Proof</u>

Let's take $C$ and assume that it is $C = SJS^{-1}$ is in Jordan canonical form.

If $e^K = J$, $Se^KS^{-1} = SJS^{-1} = C$.

$\hookrightarrow e^{SKS^{-1}}$

We only do it for $C = J = \lambda I + N$, $N^m = 0$, $0 \leq m < n$. Since $C$ is non-singular then $\lambda \neq 0$, so we write this $C = \lambda(I + \frac{1}{\lambda}N)$

Remember the Taylor series of $\log(1+t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} t^k$, $|t| <$

Formally, we define

$$\log\left(I + \frac{1}{\lambda}N\right) = \sum_{k=1}^{m-1} \frac{(-1)^{k-1}}{k\lambda^k} N^k$$

$B = \log(C) = \log(\lambda I \cdot (I + \frac{1}{\lambda}N)) = \log(\lambda I) + \log(I + \frac{1}{\lambda}N) = \log(\lambda) I + \log(I + \frac{1}{\lambda}N)$

$$B = \log(\lambda) I + \sum_{k=1}^{m-1} \frac{(-1)^{k-1}}{k\lambda^k} N^k. \qquad e^B = \lambda \cdot \exp\left(\sum_{k=1}^{m-1} \frac{(-1)^{k-1}}{k\lambda^k} N^k\right) = C$$

↳ Verify that this is equal to $I + \frac{1}{\lambda}$

We notice that the eigenvalues $C^2$ are the squares of the eigenvalues of

So $B$ is in fact real.

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Proof

If we have Jordan Blocks of the form

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad r > 0$$

We use the formula for the exponential matrix

$$e^{(\log r) I + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\theta} = r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$J = \begin{pmatrix} D & I & & \\ & D & I & \\ & & \ddots & \\ & & & D\,I \end{pmatrix}, \quad D = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

Now we know how to do of matrices in JCF. //

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$0 \leq m < n$. Since $C$ is non-singular then $\lambda \neq 0$, so we write this $C = \lambda(I + \frac{1}{\lambda}N)$

Remember the Taylor series of $\log(1+t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} t^k$, $|t| < 1$

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## Notes

- We proved the theorem by using Prop.
  There are other more direct ways to do it.
  The proposition can be extended.
- Many of the techniques can be extended to infinite dims.

$$\log\left(I + \tfrac{1}{\lambda}N\right) = \sum_{k=1}^{m-1} \frac{(-1)^k}{k\lambda^k} N^k$$

apply this to operators in Banach $\mathcal{L}(E)$

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- Liouville's Formula

$\Phi(t) \to \dot{x}=A(t)x$

$$\det\Phi(t)=\det\Phi(t_0)\exp\int_{t_0}^{t}\operatorname{tr}A(s)\,ds$$

Corollary (Lemma 2.1.31 – Kapitula & Promislow 10/

Let $\{\mu_j\}_{j=1}^{n}$ & $\{\lambda_j\}_{j=1}^{n}$ are the Floquet exponents and multipliers associated to $\dot{x}=A(t)x$, $A(t+T)=A(t)$ $\forall t\in\mathbb{R}$

It holds that

$$\prod_{j=1}^{n}\lambda_j=e^{\int_0^T \operatorname{tr}(A(s))ds},\qquad \sum_{j=1}^{n}\mu_j=\frac{1}{T}\int_0^T \operatorname{tr}(A(s))\,ds$$

$\lambda_j$ ↳ Floquet multipliers

$\boxed{e^{T\mu_j}=\lambda_j}$

Volunteers to prove this in class.

Bonus (Extra credit).

Example

$$A(t)=\begin{pmatrix}-1+\frac{3}{2}\cos^2 t & 1-\frac{3}{2}\sin t\cos t\\ -1-\frac{3}{2}\sin t\cos t & -1+\frac{3}{2}\sin^2 t\end{pmatrix}$$

- Eigenvalues $\longrightarrow -\frac{1}{4}\pm\frac{\sqrt{7}}{4}i$ but $x(t)=e^{t/2}\begin{pmatrix}-\cos t\\ \sin t\end{pmatrix}$
- The $\operatorname{tr}(A(t))=-\frac{1}{2}$

$$\lambda_1\cdot\lambda_2=e^{\int_0^T -\frac{1}{2}ds}=e^{-T/2}<1$$

$$\mu_1+\mu_2=-\frac{1}{2}$$

From we know that $\lambda_1>1$, thus $\lambda_2<1$

So the other solution is stable $x=0$.

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$\Phi(t) = Q(t)e^{Rt}$.

We will prove the thm using logs of matrices.

- Liouville's Formula

$\Phi(t) \to \dot{x} = A(t)x$

$\det \Phi(t) = \det \Phi(t_0) \exp \int_{t_0}^{t} tr\, A(s)\, ds$

Let's look at the relationship between the periodic case and the constant coefficient case.

Prop (2.89 - Chicone '06)

If the priciple fundamental matrix solution of the $T$-periodic ODE $\dot{x} = A(t)x$ is given by $P(t)e^{tB}$, where $P(t)$ is periodic then the change of coordinates $x = P(t)y$ transforms the system into a constant coefficient system.

Proof

Notice that $x(t) = \underbrace{P(t)e^{tB}}_{P.F.M.S.}\overbrace{\phantom{x}}^{y(t)} x(0)$, then $P(0) = I$.

Then $y(t) = e^{tB}x(0)$ and $\dot{y} = By$. //

$\dot{x} = A(t)x \xrightarrow{x = P(t)y} \dot{y} = By$.

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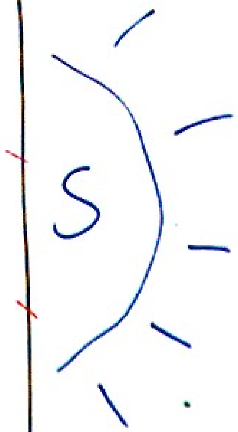
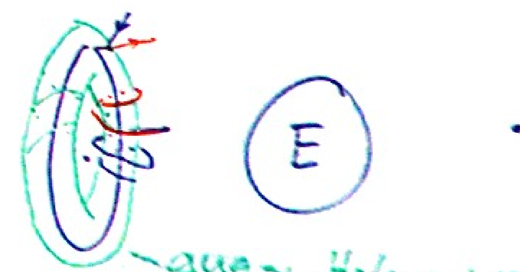
There is also an analogous theory for quasi-periodic problems.
We are no going to go into that. (Unless someone is interested).
Or go to pages 199-200 (Chicone's '06).

Related to Invariant tori.

Space mission design.




Normally Hyperbolic Invariant Manifold.




quasi-Halo orbit

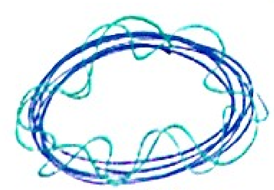
Invariant tori

quasi-periodic dynamics.

# Inhomogeneous / forced linear systems

We will talk about ODE's of the form.

$$\dot{x} = \underbrace{A(t)x}_{\text{linear part}} + \underbrace{f(t)}_{\text{forcing term.}}$$




For instance assume that $A(t)=A$ is constant coefficient.
A is an $n\times n$ matrix

$$\dot{x} = Ax + f(t), \quad x(t_0)=x_0$$

In undergrad ODE:

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)} f(s)\,ds$$

$$\underbrace{e^{-At}\dot{x}(t) - e^{-At}Ax(t)} = e^{-At}f(t)$$

$$\int_{t_0}^{t} \frac{d}{ds}\left(e^{-As}x(s)\right)ds = \int_{t_0}^{t} e^{-sA} f(s)\,ds$$

$$e^{tA}\left(e^{-tA}x(t) - e^{-t_0A}x_0 = \int_{t_0}^{t} e^{-sA} f(s)\,ds\right)$$

$$x(t) - e^{(t-t_0)A}x_0 = \int_{t_0}^{t} e^{(t-s)A} f(s)\,ds$$

$$\dot{x} = A(t)x + f(t), \quad x(t_0)=x_0, \quad \dot{\Phi}(t) = A(t)\Phi(t), \quad \Phi(t_0)=I$$

$$\Phi(t)^{-1}\dot{x}(t) - \Phi(t)^{-1}A(t)x(t) = \Phi(t)^{-1}f(t)$$

$$\int_{t_0}^{t} \frac{d}{ds}\left(\Phi(s)^{-1}x(s)\right)ds = \int_{t_0}^{t} \Phi(s)^{-1}f(s)\,ds$$

$$\Phi(t)^{-1}x(t) - \Phi^{-1}(t_0)x_0 = \int_{t_0}^{t} \Phi(s)^{-1}f(s)\,ds$$

$$x(t) - \Phi(t)\Phi^{-1}(t_0)x_0 = \Phi(t)\int_{t_0}^{t} \Phi(s)^{-1}f(s)\,ds$$

Gen. soln. $x(t) = \Phi(t)\underbrace{\Phi^{-1}(t_0)}_{=\,\mathrm{Id}}x_0 + \Phi(t)\int_{t_0}^{t}\Phi(s)^{-1}f(s)\,ds$