Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

Renato Calleja, 28 de febrero de 2024

Linear constant coefficient case

$$\dot{X} = Ax, \quad X(t_0) = x_0 \qquad (*)$$

$$X(t) = e^{(t-t_0)A} x_0$$

Hyperbolic matrices $\mathrm{Re}\{\sigma(A)\} \neq 0$

What happens when the matrix is periodic?

$$\dot{X} = A(t)x, \quad A(t+T) = A(t), \ \forall t \in \mathbb{R} \quad (**)$$
$$x(t_0) = x_0$$

- Floquet Theory. (Due to Gaston A. M. Floquet)

There is a transformation that reduces $(**)$ to $(*)$

Note. Computing the eigenvalues of $A(t)$ in $(**)$ is not useful.

Example (L. Marcus & Hidehiko Yamabe)

$$\dot{X} = A(t)X, \quad A(t+T) = A(t), \quad X \in \mathbb{R}^2$$

$$A(t) = \begin{pmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\sin t\cos t \\ -1 - \frac{3}{2}\sin t\cos t & -1 + \frac{3}{2}\sin^2 t \end{pmatrix}$$

What is T (minimal period)?

$T = \pi$ (not 2π)

Verify that $A(t+\pi) = A(t)$

$$\lambda = \frac{tr(A(t)) \pm \sqrt{tr^2(A(t)) - 4\det(A(t))}}{2}$$

↳ This comes from $\det(A(t) - \lambda I) = 0$

$$tr(A(t)) = -2 + \tfrac{3}{2}(1) = -\tfrac{1}{2}$$

$$\det(A(t)) = (-1 + \tfrac{3}{2}\cos^2 t)\cdot(-1 + \tfrac{3}{2}\sin^2 t) - (1 - \tfrac{3}{2}\sin t\cos t)\cdot(-1 - \tfrac{3}{2}\sin t\cos t)$$
$$= 1 - \tfrac{3}{2}(1) + \cancel{\tfrac{9}{4}\cos^2 t\sin^2 t} - (-1 + \cancel{\tfrac{9}{4}\sin^2 t\cos^2 t} - \tfrac{3}{2}(\cancel{\sin t\cos t} - \cancel{\sin t\cos t}))$$
$$= 2 - \tfrac{3}{2} = \tfrac{1}{2}$$

$$\lambda = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 2}}{2} = -\frac{1}{4} \pm \frac{1}{2} i \frac{\sqrt{7}}{2} = \frac{-1 \pm i\sqrt{7}}{4}$$

$$\boxed{\operatorname{Re}(\lambda) = -\tfrac{1}{4}}$$

$$\operatorname{Im}(\lambda) = \pm\left(\frac{\sqrt{7}}{4}\right)$$

The $X=0$ solution "should" be stable.

It is not stable.

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$$\begin{aligned}\det(A(t)) &= (-1 + \tfrac{3}{2}\cos^2 t)\cdot(-1 + \tfrac{3}{2}\sin^2 t) \\ &\quad - (1 - \tfrac{3}{2}\sin t\cos t)\cdot(-1 - \tfrac{3}{2}\sin t\cos t) \\ &= 1 - \tfrac{3}{2}(1) + \cancel{\tfrac{9}{4}\cos^2 t\sin^2 t} \\ &\quad - (-1 + \cancel{\tfrac{9}{4}\sin^2 t\cos^2 t} - \tfrac{3}{2}(\cancel{\sin t\cos t} - \cancel{\sin t\cos t})) \\ &= 2 - \tfrac{3}{2} = \tfrac{1}{2}\end{aligned}$$

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$\boxed{\mathrm{Re}(\lambda) = -\frac{1}{4}}$ The $x=0$ solution "should" be stable.

$\mathrm{Im}(\lambda) = \pm\left(\frac{\sqrt{7}}{4}\right)$ It is not stable.

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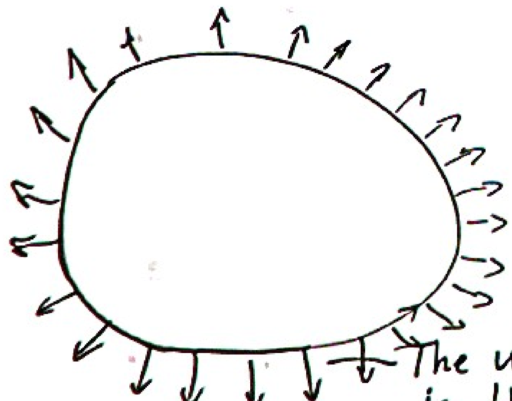
Notice $x(t) = e^{t/2}(-\cos t, \sin t)$ is a solution. (HW).

This grows as $t \to +\infty$.

The eigenvalues didn't predict the stability.

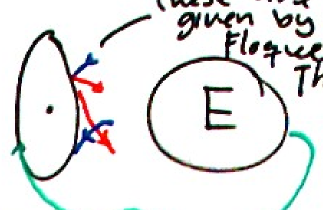
$$A(t) = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \quad \lambda(t) = \frac{\mathrm{tr}(A(t)) \pm \sqrt{\mathrm{tr}^2 A(t) - 4\det(A(t))}}{2}$$

Floquet



The unstable manifold to 1st order is the direction given by Floquet's theorem.

CR3BP + Solar Sail.



These are given by Floquet's Thm.

Station keeping.

James Webb

Floquet

Any system of the form $\underbrace{\dot{X}=A(t)X}$, $A(t+T)=A(t)$

$\rightarrow$ there is a unique transformation taking $\rightarrow$ $\dot{y}=By$ constant coefficient.

Thm (2.83 Chicone '06)

If $\Phi(t)$ is a fundamental matrix solution of

$\dot{X}=A(t)X$, $X\in\mathbb{R}^n$, $A(t+T)=A(t)$

then for all $t\in\mathbb{R}$

$$\Phi(t+T)=\Phi(t)\,\Phi^{-1}(0)\,\Phi(T)$$

In addition, there is a matrix B, possibly complex such that $e^{TB}=\Phi^{-1}(0)\,\Phi(T)$

and a T-periodic matrix function $P(t)$, possibly complex such that $\Phi(t)=P(t)e^{Bt}$ for all t.

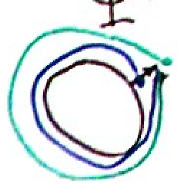
Also, there is a real matrix R and a $2T$-periodic function $Q(t)$ so that $\Phi(t)=Q(t)e^{Rt}$

- $\Phi(t) = P(t)e^{tB}$ is called the Floquet Normal Form.

(Normal form means that we write an expression in its "simplest" form.)

- The eigenvalues μ of B are called Floquet exponents associated to $A(t)$, and for each such μ, we have $\lambda \stackrel{\text{def}}{=} e^{\mu T}$ are called Floquet multipliers.
- If x is an initial condition, then evolving that vector over time amounts to

$$\Phi(T)x = P(T)e^{TB}x = P(0)e^{TB}x$$



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. and a T-periodic matrix function $P(t)$, possibly complex such that $\Phi(t) = P(t)e^{Bt}$, for all t.

Also, there is a real matrix R and a $2T$-periodic function $Q(t)$ so that $\Phi(t) = Q(t)e^{Rt}$

$\lambda = e^{\mu T}$ — Floquet exponents $\exp: \mathbb{C} \to \mathbb{C}$

— Floquet multipliers

- Note that Floquet exponents are not unique

since $\lambda = e^{\mu T + 2\pi i k}$, $k \in \mathbb{Z}$

$$e^{2\pi i k} = 1$$

That means that if μ is a Floquet exponent,

then so is $\mu + 2\pi i k/T$ $\quad e^{T(\mu + 2\pi i k/T)} = e^{T\mu} e^{2\pi i k} = \lambda$

$\forall k \in \mathbb{Z}$

That one of the reasons why some people prefer Floquet multipliers.