Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

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Last class

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Prop

A an $n \times n$ matrix, then e^{tA} is a matrix with components that are finite sums of the form

$$p(t)\, e^{at} \sin bt \quad , \quad p(t) e^{at} \cos bt$$

where $\lambda = a + ib$ is an eigenvalue of A and $p(t)$ is a polynomial of dimension $n-1$.

If $a < 0$ then e^{at} is exponential decay so we have stability.

$\|e^{tA}\| \longrightarrow 0$ if all the real parts of the eigenvalues are negative.

Asymptotic stability theorem

Thm (2.61) (Chicone '06)
A $n \times n$ matrix. Then the following are equivalent.
A $n \times n$ matrix and a real number $\lambda > 0$

1) There is a norm $\|\cdot\|_a$ on $\mathbb{R}^n$ and a real number $\lambda>0$ such that for all $v \in \mathbb{R}^n$ and all $t \geq 0$.
$$\|e^{tA} v\|_a \leq e^{-\lambda t} \|v\|_a$$

2) If $\|\cdot\|_g$ is any norm on $\mathbb{R}^n$, then there is a constant $C>0$ and a real number $\lambda>0$ such that for all $v \in \mathbb{R}$ and $t \geq 0$
$$\|e^{tA} v\|_g \leq C e^{-\lambda t} \|v\|_g$$

3) Every eigenvalue of A has negative real part.

∘ Note we will show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$

Cor
If every eigenvalue of A has negative real part then the zero solution is asymptotically stable.

Proof

$1 \Rightarrow 2$.
Recall that all the norms of $\mathbb{R}^n$ are equivalent

$$K_1 \|x\|_g \leq \|x\|_a \leq K_2 \|x\|_g$$
$$\|e^{tA} v\|_g \leq \frac{1}{K_1} \|e^{tA} v\|_a \leq \frac{1}{K_1} e^{-\lambda t} \|v\|_a \leq \frac{K_2}{K_1} e^{-\lambda t} \|v\|_g$$
$$\|e^{tA} v\|_g \leq \underbrace{\frac{K_2}{K_1}}_{\text{this is the constant } C \text{ from}} e^{-\lambda t} \|v\|_g$$

$2 \Rightarrow 3$ We prove this by contradiction.
Supose there is an eigenvalue of A that doesn't have negative real part. Namely $\lambda = a + i\beta$ $a \geq 0$.
Then there is a solution $\gamma(t) = e^{at}[(\cos(\beta t)u - \sin$
So along the direction of γ

Asymptotic stability theorem

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<u>Proof</u>

$3 \Rightarrow 1$

- Since there are a finite # of eigenvalues, there exists $\tilde{\lambda}$ so that all the real parts of the eigenvalues is less than $-\tilde{\lambda}$.

From the previous proposition, we know that the solu are of the form $e^{\alpha t}[p(t)\cos(\beta t)u - q(t)(\sin(\beta t))w]$

p, q are polynomials of degree at most $n-1$.

That means

$$\|e^{tA} v\| \leq M|t|^{n-1} e^{-\tilde{\lambda} t} \|v\|$$

- $\|\cdot\|$ is the usual Eucledian norm.

There exsts some time τ and λ such that, $\forall t \geq \tau$

$$M|t|^{n-1} e^{-\tilde{\lambda} t} \|v\| \leq e^{-\lambda t} \|v\|$$

Asymptotic stability theorem

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A $n\times n$ matrix. Then the following are equivalent.

1) There is a norm $\|\cdot\|_a$ on $\mathbb{R}^n$ and a real number $\lambda>0$ such that for all $v\in\mathbb{R}^n$ and all $t\geq 0$.
$$\|e^{tA}v\|_a \leq e^{-\lambda t}\|v\|_a$$

2) If $\|\cdot\|_g$ is any norm on $\mathbb{R}^n$, then there is a constant $C>0$ and a real number $\lambda>0$ such that for all $v\in\mathbb{R}^n$ and $t\geq 0$
$$\|e^{tA}v\|_g \leq Ce^{-\lambda t}\|v\|_g$$

3) Every eigenvalue of A has negative real part.

- Note we will show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$

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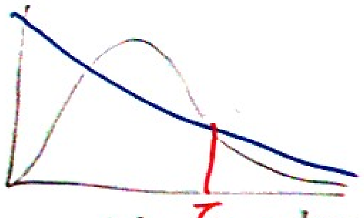
Proof



$3\Rightarrow 1$
$$\|v\|_a = \int_0^\tau e^{\lambda s}\|e^{sA}v\|\,ds$$
Let's see that this norm has the right properties
$$\|e^{tA}v\|_a = \int_0^\tau e^{\lambda s}\|e^{(s+t)A}v\|\,ds$$
for any t we write $t = m\tau + T$, with $m\in\mathbb{N}$, $0\leq T$

$$\|e^{tA}v\|_a = \int_0^{\tau-T} e^{\lambda s}\|e^{(s+t)A}v\|\,ds + \int_{\tau-T}^{\tau} e^{\lambda s}\|e^{(s+t)A}v\|\,ds$$
$$= \int_0^{\tau-T} e^{\lambda s}\|e^{m\tau A}e^{(s+T)A}v\|\,ds + \int_{\tau-T}^{\tau} e^{\lambda s}\|e^{(m+1)\tau A}e^{\ldots}$$
$s+T = m\tau+u$ $\qquad$ $s+T-\tau = (m+1)\tau+u$
$$= \int_T^\tau e^{\lambda(u-T)}\|e^{(m\tau+u)A}v\|\,du + \int_0^T e^{\lambda(u-T+\tau)}\|e^{((m+1)\tau+\ldots}$$
$$\leq \int_T^\tau e^{\lambda(u-T)}e^{-\lambda m\tau}\|e^{uA}v\|\,du + \int_0^T e^{\lambda(u-T+\tau)}e^{-(m+1)\lambda\tau}\|e\ldots$$
$$= \int_0^\tau e^{\lambda u}e^{-\lambda\underbrace{(m\tau+T)}_{t}}\|e^{uA}v\|\,du = e^{-\lambda t}\int_0^\tau$$

Asymptotic stability theorem: Hyperbolic matrices

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Comments

- The corollary $\Rightarrow$ Asymtotic stability.
- There is something to say about hyperbolic syste

A if all λ's have positive real part.
by the theorem exchange t for $-t$
we get exponetial decay in the past.

$\rightarrow$ A hyperbolic matrix is a matrix with eigenvalu that have either positive or negative real parts,

- but no zero real part.

$\Rightarrow$ There are submatrices A^u, A^s where

A^u – subblock with all the directions corresp to eigenval with positive real part.

A^s – " " " " negative real part.

These matrices define e^{tA^u} & e^{tA^s}
unstable stable.

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Hyperbolic matrix

$$A = \begin{pmatrix} A^u & 0 \\ 0 & A^s \end{pmatrix}$$

rearranging the directions.

$$e^{tA} = e^{t\begin{pmatrix} A^u & 0 \\ 0 & A^s \end{pmatrix}} = \begin{pmatrix} \overbrace{e^{tA^u}}^{\text{decays in the past}} & 0 \\ 0 & \underbrace{e^{tA^s}}_{\text{decays in the future}} \end{pmatrix}$$

$\dot{x} = A(t)x$, where $A(t)$ is periodic.