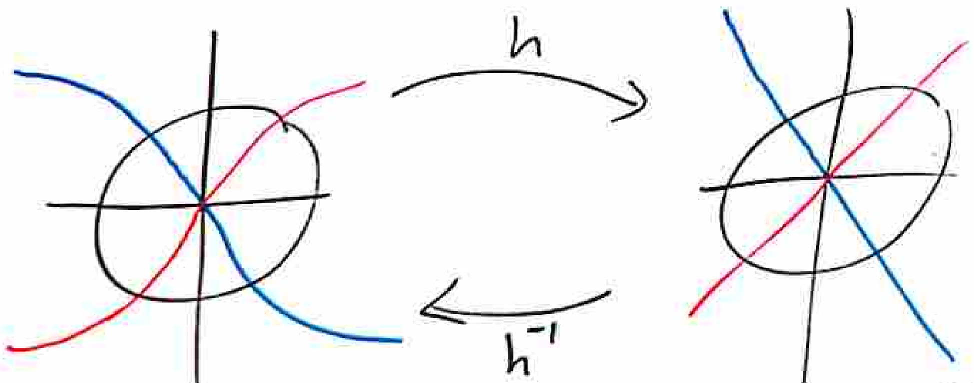
Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

Renato Calleja, 21 de marzo de 2024



Stable, unstable and center manifolds

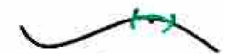
A manifold is a topological space that is locally homeomorphic to $\mathbb{R}^m$. (m = dim of the manifold)
(diffeomorphic)

"It's a space that locally looks like $\mathbb{R}^m$"

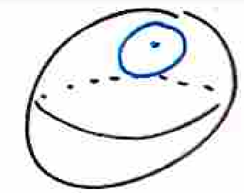
Informally an m-dimensional manifold embedded in $\mathbb{R}^{L>m}$ is a subset $\mathcal{M} \subset \mathbb{R}^L$ s.t. there exists a neighborhood of $x \in \mathcal{M}$ that look like $\mathbb{R}^m$.

Ex

point
(a fixed point is an invariant manifold)



line
(a flow line is an inv. manifold).

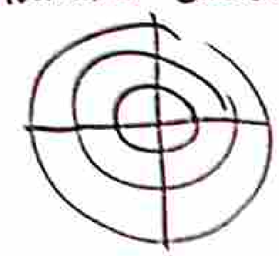


sphere

torus

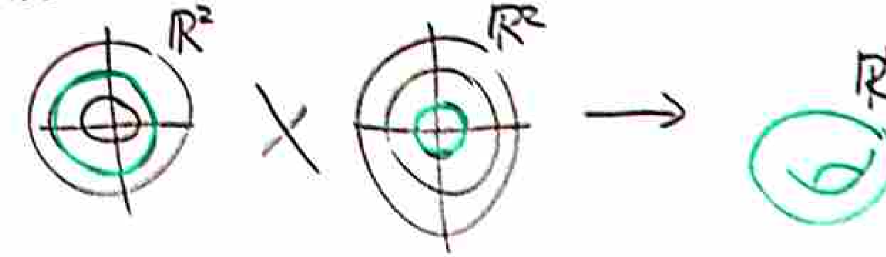
Periodic orbits - they are embedded invariant manifolds - Invariant circles.

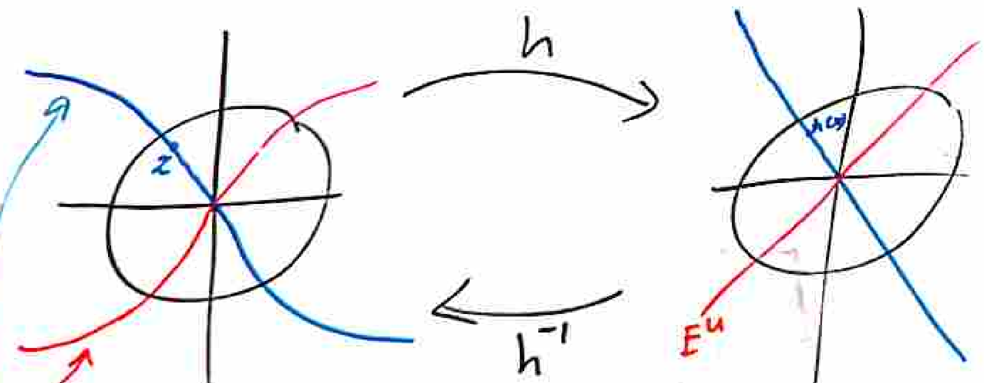
harmonic osallator $\longrightarrow$



2 uncoupled harmonic oscillators

$e^{\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} t}$





Stable, unstable manifolds

Def The stable manifold of a fixed point p for an autonomous dynamical system with flow $\varphi(t;x)$ is the set of all points x for which the flow is defined and

$$\lim_{t\to\infty} \varphi(t;x) = p$$

The unstable manifold " $\lim_{t\to-\infty} \varphi(t;x) = p$

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point (a fixed point is an invariant manifold)

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sphere

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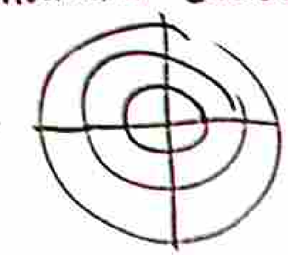
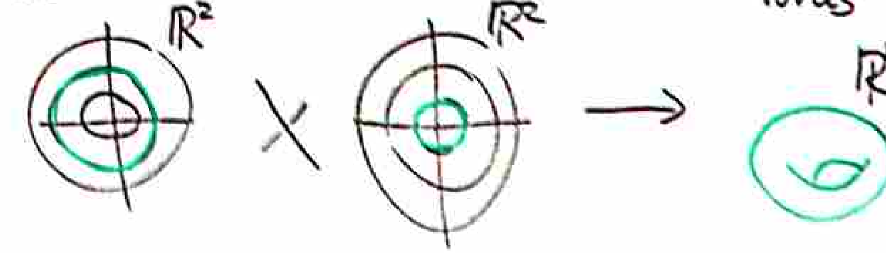


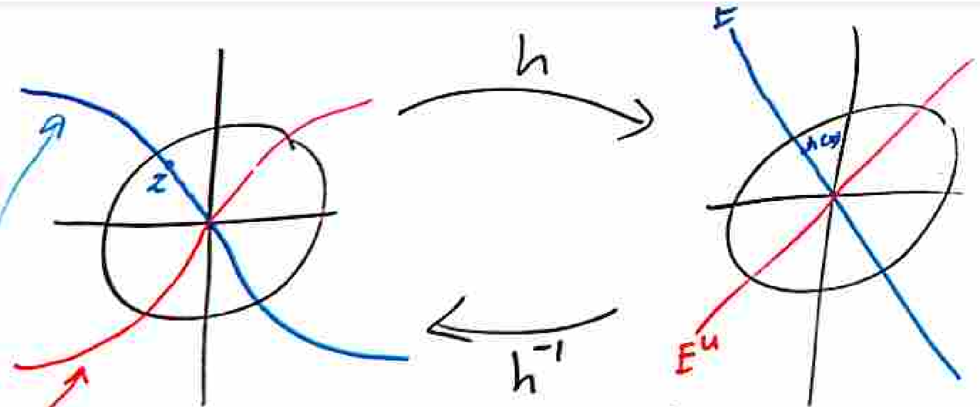
Klein bottle $\mathbb{R}^4$

Torus

2 uncoupled harmonic oscillators

$e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t}$





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Remember

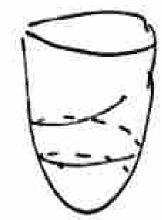
$$\dot{x} = -y - x^3$$
$$\dot{y} = x - y^3$$

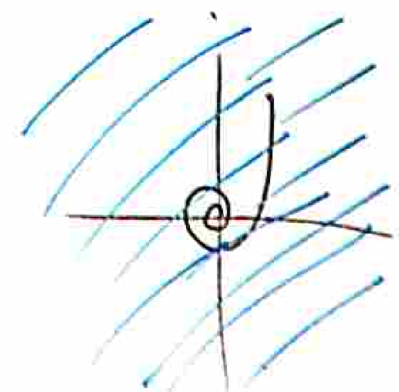
The origin is not hyperbolic

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

However $V(x,y) = x^2 + y^2$ is a Lyapunov function

The origin is globally asymptotically stable.

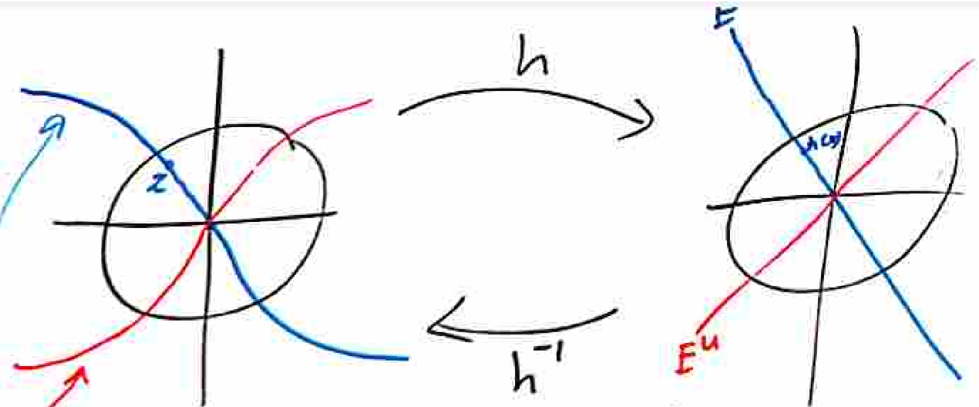




$\mathbb{R}^2$ is the stable manifold of the origin

What is the unstable manifold?

The origin.



Stable, unstable manifolds

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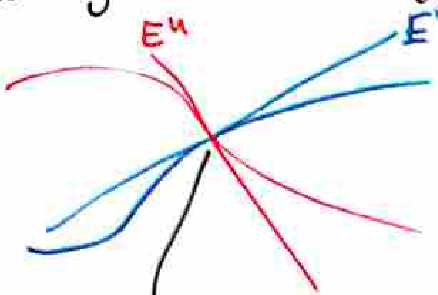
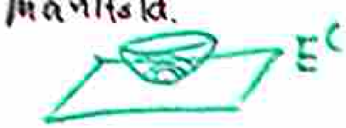
We will show that that stable and unstable manifold always exist in a neighborhood of a hyperbolic fixed point.

And the will be tangent to the the stable E^s and unstable space E^u respectively at the origin.



When the linearization has eigenvalues with real part equal to zero there is an invariant space E^c there is an invariant manifold for the nonlinear flow that is tangent to this space, the center manifold.



Example

$$\frac{dx}{dt} = 2x - (2+y)e^{y} \qquad (*)$$
$$\frac{dy}{dt} = -y$$

In HW#1 you proved that the level sets of

$$H(x,y) = y^2 e^{y} - xy^2 = c$$

are invariant under the flow of $(*)$

A fixed point

$-y = 0 \Rightarrow y = 0$

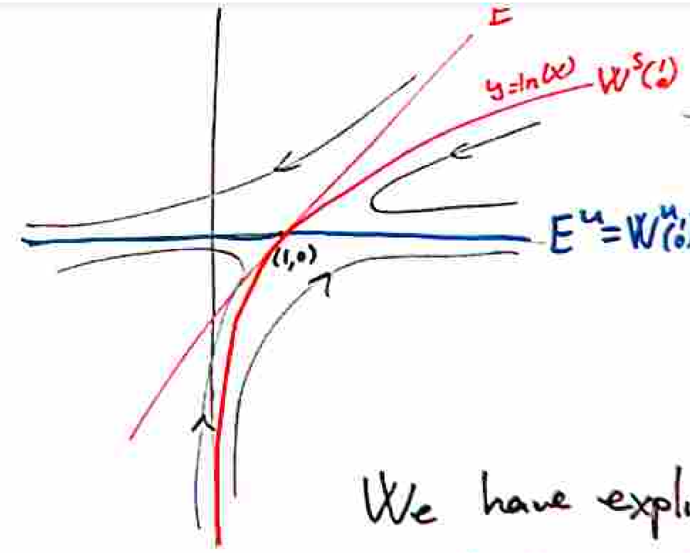
$+2x - (2+y)e^{y} = 0 \Rightarrow +2x - (2)1 = 0$

$2(x-1) = 0, \; x = 1$

$(x,y) = (1,0)$ is a fixed point.

$$Df\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix} 2 & -e^{y} - (2+y)e^{y} \\ 0 & -1 \end{pmatrix}, \quad Df\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix}$$

$$\left.\begin{aligned} E^{s} &= \operatorname{span}\left\{\begin{pmatrix}1\\-1\end{pmatrix}\right\} \\ E^{u} &= \operatorname{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\} \end{aligned}\right\} \text{linear algebra.}$$



Level set of the fixed point.

$H(1,0) = 0$

$y^2 e^{y} - x y^2 = 0$ is invariant.

$y^2(e^{y} - x) = 0$

Either $y = 0$
or $x = e^{y} \Rightarrow y = \ln(x)$

We have explicitly the

stable manifold $W^{s}\binom{1}{0}$

unstable manifold $W^{u}\binom{1}{0}$

Def

fixed point

$$W^{s}(\tilde{x}) = \{x \in \mathbb{R}^{L} \mid \lim_{t\to\infty} \varphi_t(x) = \tilde{x}\} \quad \text{stable manifold}$$

$$W^{u}(\tilde{x}) = \{x \in \mathbb{R}^{L} \mid \lim_{t\to-\infty} \varphi_t(x) = \tilde{x}\} \quad \text{unstable manifold}$$

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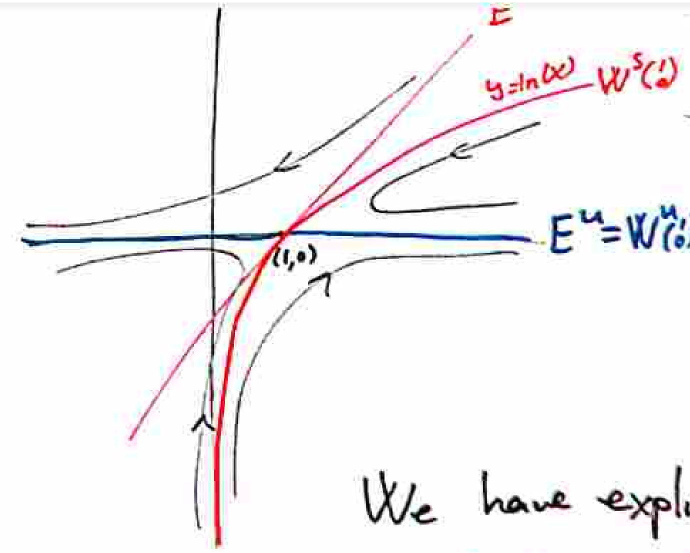

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