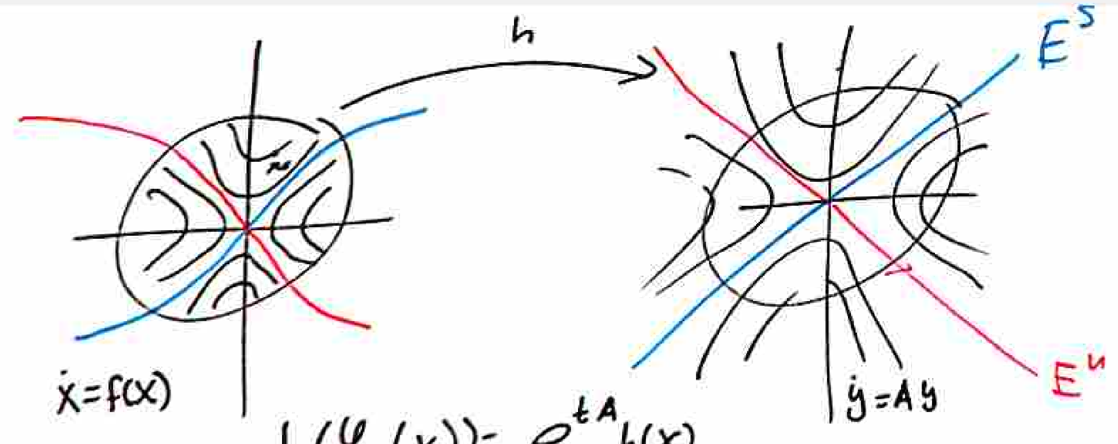
Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

Renato Calleja, 20 de marzo de 2024

Proof of the Grobman-Hartman Theorem



$h(\varphi_t(x)) = e^{tA} h(x)$

Pf

$f \in C^1(U)$, $f(0)=0$ (otherwise we translate)

$A = Df(0)$, $\dot y = Ay$ hyperbolic.

$A = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$ P has eigenvalues with neg. real part.
Q " " " pos. real part.

$X(t) = \varphi_t(X_0) = \begin{pmatrix} w(t; w_0, v_0) \\ v(t; w_0, v_0) \end{pmatrix}$, $X_0 = \begin{pmatrix} w_0 \\ v_0 \end{pmatrix}$ $w_0 \in E^s$, $v_0 \in E^u$, $\mathbb{R}^n = E^s \oplus E^u$

Define

$\tilde W(w_0, v_0) = w(1; w_0, v_0) - e^P w_0$

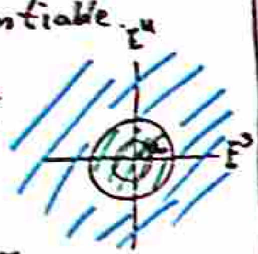
$\tilde V(w_0, v_0) = v(1; w_0, v_0) - e^Q v_0$ (abusing notation).

then $\tilde W(0) = \tilde V(0) = D\tilde W(0) = D\tilde V(0) = 0$

Since $f \in C^1$ then $\tilde W$ & $\tilde V$ are continuously differentiable.

Thus $\|D\tilde W(w_0, v_0)\| \le a$ with $|w_0|^2 + |v_0|^2 \le s_0^2$

$\|D\tilde V(w_0, v_0)\| \le a$



Let the smooth functions $W(w_0, v_0)$ & $V(w_0, v_0)$ to be equal to $\tilde W$ & $\tilde V$ inside of $|w_0|^2 + |v_0|^2 \le (s_0/2)^2$ and zero outside of $|w_0|^2 + |v_0|^2 \ge s_0^2$

By the mean value theorem

$|W(w_0, v_0)| \le a\sqrt{|w_0|^2 + |v_0|^2} \le a(|w_0| + |v_0|)$

$|V(w_0, v_0)| \le a(|w_0| + |v_0|)$

$\forall (w_0, v_0) \in \mathbb{R}^n$.

Pf (Cont)

$B = e^P$ & $C = e^Q$

Normalization of HW #2, Question #2.

$B = \lambda I + N$ $\exists S$ s.t. $S^{-1} B S = \lambda I + \varepsilon N$

then

$b = \|B\| < 1$, $c = \|C^{-1}\| < 1$

For $(w, v) \in \mathbb{R}^n$ define

$$L(w,v) = \begin{pmatrix} Bw \\ Cv \end{pmatrix} \quad \& \quad T(w,v) = \begin{pmatrix} Bw + W(w,v) \\ Cv + V(w,v) \end{pmatrix}$$

Notice $L(x) = e^A x = \varphi_1(x)$, $T(x) = \varphi_1(x)$

Let's construct a homeo $h: \mathcal{U} \to \mathcal{V} \subseteq \mathbb{R}^n$

$$h \circ T = L \circ h, \quad h(x) = \begin{pmatrix} \Phi(w,v) \\ \Psi(w,v) \end{pmatrix}$$

So is equivalent to

$$\begin{pmatrix} \Phi(Bw + W(w,v), Cv + V(w,v)) \\ \Psi(Bw + W(w,v), Cv + V(w,v)) \end{pmatrix} = h \circ T = L \circ h = \begin{pmatrix} B\Phi(w,v) \\ C\Psi(w,v) \end{pmatrix}$$

$$B\Phi(w,v) = \Phi(Bw + W(w,v), Cv + V(w,v))$$

$$\Psi(w,v) = C^{-1}\Psi(Bw + W(w,v), Cv + V(w,v))$$

Take this equation and do succesive approximations

$$\Psi_0(w,v) = v$$

$$\Psi_{k+1}(w,v) = C^{-1}\Psi_k(Bw + W(w,v), Cv + V(w,v))$$

It is easy to prove that these functions are continuous and

$$\Psi_k(w,v) = v \qquad |w| + |v| \geq 2s_0$$

$\boxed{\text{Volunteers?}}$

Now, let's prove that for $j = 1, 2, 3, \dots$

$$|\Psi_j(w,v) - \Psi_{j-1}(w,v)| \leq M r^j (|w| + |v|)^\delta$$

where $r = c(2\max(a,b,c))^\delta$ with $\delta \in (0,1]$

so that $r < 1$, $M = ac(2s_0)^{1-\delta}/r$

We prove it by induction.

Pf (Cont)

We prove $(*)$ by induction

$j=1$, $|\Psi_1(w,v) - v| = |C^{-1}(Cv+V(w,v)) - v| = |C^{-1}V(w,v)|$

$\leq \|C^{-1}\| \; |V(w,v)| \leq c\, a(|w|+|v|) \leq Mr\,(|w|+|v|)^{\delta}$ $\forall \delta\in[0,1]$

since $V(w,v)=0$ if $|w|+|v|\geq 2s_0$

Now we assume it true for $j=1,\dots,k$ and for $k+1$

$|\Psi_{k+1}(w,v) - \Psi_k(w,v)| \leq \|C^{-1}\| \, |\Psi_k(\square) - \Psi_{k-1}(\square)|$

$|W| \leq a(|w|+|v|)$
$|V| \leq a(|w|+|v|)$

$\leq c M r^k (|Bw + W(w,v)| + |Cv + V(w,v)|)^{\delta}$

$\leq c M r^k (b|w| + c|v| + 2a(|w|+|v|))^{\delta}$

$\leq c M r^k (2\max(a,b,c))^{\delta} (|w|+|v|)^{\delta}$

$\leq M r^{k+1} (|w|+|v|)^{\delta}$

Then Ψ_k is a Cauchy sequence that converges to Ψ in a Banach space and Ψ satisfies

$$C\Psi(w,v) = \Psi(\square)$$

$B\Phi(w,v) = \Phi(Bw+W(w,v), Cv+V(w,v))$

$\Psi(w,v) = C^{-1}\Psi(Bw+W(w,v), Cv+V(w,v))$

Take this equation and do successive approximations

$\Psi_0(w,v) = v$

$\Psi_{k+1}(w,v) = C^{-1}\Psi_k(\overbrace{Bw+W(w,v), Cv+V(w,v)}^{(\square)})$

It is easy to prove that these functions are continuous and

$\Psi_k(w,v) = v \qquad |w|+|v| \geq 2s_0$ $\quad$ Volunteers?

Now, let's prove that for $j=1,2,3,\dots$

$$|\Psi_j(w,v) - \Psi_{j-1}(w,v)| \leq M r^j (|w|+|v|)^{\delta} \qquad (*)$$

where $r = c(2\max(a,b,c))^{\delta}$ with $\delta\in[0,1]$ so that $r<1$, $M = ac(2s_0)^{1-\delta}/r$

We prove it by induction.

Pf (Cont)

We prove $(*)$ by induction

$j=1$, $|\Psi_1(w,v) - v| = |C^{-1}(Cv + V(w,v)) - v| = |C^{-1}V(w,v)|$

$\leq \|C^{-1}\| \, |V(w,v)| \leq c\, a(|w|+|v|) \leq Mr \, (|w|+|v|)^{\delta} \; \forall \delta \in [0,1]$

since $V(w,v) = 0$ if $|v|+|v| \geq 2s_0$

Now we assume it true for $j = 1, \dots, k$ and for $k+1$

$|\Psi_{k+1}(w,v) - \Psi_k(w,v)| \leq \|C^{-1}\| \, |\Psi_k(\square) - \Psi_{k-1}(\square)|$

$|W| \leq a(|w|+|v|)$
$|V| \leq a(|w|+|v|)$

$\leq c M r^k (|Bw + W(w,v)| + |Cv + V(w,v)|)^{\delta}$

$\leq c M r^k (b|w| + c|v| + 2a(|w|+|v|))^{\delta}$

$\leq c M r^k (2 \max(a,b,c))^{\delta} (|w|+|v|)^{\delta}$

$\leq M r^{k+1} (|w|+|v|)^{\delta}$

Then Ψ_k is a Cauchy sequence that converges to Ψ in a Banach space and Ψ satisfies

$$C\Psi(w,v) = \Psi(\square)$$

$B\Phi(w,v) = \Phi(Bw + W(w,v), Cv + V(w,v))$

For the first we write

$B^{-1}\Phi(w,v) = \Phi(B^{-1}w + W_1(w,v), C^{-1}v + V_1(w,v))$

W_1 & V_1 are obtained from the inverse of T

$\Phi_0(w,v) = w$, and $b^{-1} = \|B^{-1}\| < 1$

so we obtain a continuous map

$$h(w,v) = \begin{pmatrix} \Phi(w,v) \\ \Psi(w,v) \end{pmatrix}$$

which defines the homeomorphism.

$L \circ h = h \circ T$

Now, what we needed was the conjugation with the flows φ_t, ψ_t

Pf (Cont)

Now, let L^t & T^t be the one parameter families

$$L^t(x_0) = e^{tA} x_0 \quad \text{and} \quad T^t(x_0) = \varphi_t(x_0)$$

There exists a neighborhood of the origin where $h = \int_0^1 L^{-s} h T^s ds$

$$L^t \circ h = \int_0^1 L^{t-s} h T^{s-t} ds \, T^t = \int_{-t}^{1-t} L^{-s} h T^s ds \, T^t$$

$s-t=u$

$$= \left[\int_{-t}^{0} L^{-s} h T^s ds + \int_0^{1-t} L^{-s} h T^s ds \right] T^t$$

$$\int_{-t}^{0} L^{-s} h T^s ds = \int_{-t}^{0} L^{-s-1} h T^{s+1} ds = \int_{1-t}^{1} L^{-s} h T^s ds$$

$s+1=u$

$$= \int_0^1 L^{-s} h T^s ds \, T^t = h \circ T^t, \quad L^t \circ h = h \circ T^t$$

$$h \circ \varphi_t(x_0) = e^{tA} h(x_0)$$

$$B \Phi(w,v) = \Phi(Bw + W(w,v), Cv + V(w,v))$$

For the first we write

$$B^{-1} \Phi(w,v) = \Phi(B^{-1}w + W_1(w,v), C^{-1}v + V_1(w,v))$$

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$$L \circ h = h \circ T$$

Now, what we needed was the conjugation with the flows φ_t, ψ_t