Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

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Example

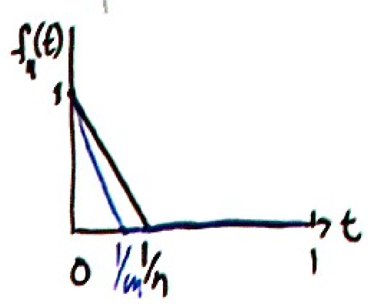
<u>Example</u>

$C([0,1],\mathbb{R})$ with the norm

$\|f\|_1 = \int_0^1 |f(t)|\,dt$ is not a Banach space.

Let's consider sequence for $n \geq 1$ define the

$$f_n(t) = \begin{cases} 1-nt & \text{if } 0 \le t \le 1/n. \\ 0 & \text{if } 1/n \le x \le 1. \end{cases}$$



$\|f_n\|_1 = 1/2n$

This is a Cauchy sequence. ✓

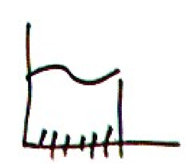
$$\|f_n - f_m\|_1 = \left|\frac{1}{2n} - \frac{1}{2m}\right| = \frac{m-n}{2nm} < \varepsilon$$

$n,m \geq N(\varepsilon)$

Pointwise

$f_n \longrightarrow f^*$

$$f^*(t) = \begin{cases} 1 & \text{if } t=0 \\ 0 & \text{if } t \neq 0 \end{cases}$$ Not continuous



$$\begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_k) \end{pmatrix} \in \mathbb{R}^k$$

$x \in \mathbb{R}^k$

$|x|_1 = \sum |x_i|$

Existence and uniqueness

Existence and uniqueness

Let $J \subseteq \mathbb{R}$, $\Omega \subseteq \mathbb{R}^n$ & $\Lambda \subseteq \mathbb{R}^k$ open sets and let's assume that $f: J \times \Omega \times \Lambda \longrightarrow \mathbb{R}^n$ is a smooth function.

Here smooth means that f is continuously differentiable.

An ordinary differential equation (ODE) (EDO) is an equation of the form

$$\dot{x} = f(t, x, \lambda) \quad (1.1)$$

The dot $(\dot{})$ corresponds to $\dot{x} = \frac{dx}{dt}$

t - independent variable
x - dependent variable $x(t)$ (state variable)
λ - parameter vector.

(1.1) is a system of ordinary differential equation.

$$\dot{x}_1 = f_1(t, x, \lambda)$$
$$\dot{x}_2 = f_2(t, x, \lambda)$$
$$\vdots$$
$$\dot{x}_n = f_n(t, x, \lambda)$$

When we are interested in different values of $\lambda \in \Lambda$ then we say that (1.1) is a family of ODEs.

Van der Pol Oscillator

Existence and uniqueness

Example

Forced Van der Pol oscilator

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = b(1-x_1^2(t))x_2(t) - w^2 x_1(t) + a\cos(\Gamma t)$$

$J = \mathbb{R}$, $x = (x_1, x_2) \in \Omega = \mathbb{R}^2$

$\Lambda = \{(a,b,w,\Gamma) : (a,b) \in \mathbb{R}^2, w > 0, \Gamma \in (0, 2\pi]\}$

$\Gamma \in \$$

$$f : \mathbb{R} \times \mathbb{R}^2 \times \Lambda \to \mathbb{R}^2$$

in components.

$$(t, x_1, x_2, a, b, w, \Gamma) \mapsto (\underbrace{x_2}_{f_1}, \underbrace{b(1-x_1^2)x_2 - w^2 x_1 + a\cos(\Gamma t)}_{f_2})$$

If $\lambda \in \Lambda$ is fixed then the solution to the differential equation (1.1) is a function

$\phi : J_0 \to \Omega$ $J = \mathbb{R}$ such that

J_0 is an open subset of

$$\frac{d\phi}{dt}(t) = f(t, \phi(t), \lambda) \qquad (1.2)$$

$$\forall t \in J_0$$

In this context the words "integral curve" "trajectory", "phase curve", also refer to solutions of the diff eq. (1.1).

We also need to talk about the image of the solution ϕ given by $\{\phi(t) \in \Omega : t \in J_0\}$

Initial Value Problems

Existence and uniqueness

When an ODE is used to describe the evolution of a state variable of a physical process we need to determine the future values of the state variable from an initial value.

The mathematical model is given by the equations

$$\dot{x} = f(t,x,\lambda)$$

$$x(t_0) = x_0 \quad \leftarrow \text{initial condition}$$

Initial value problem

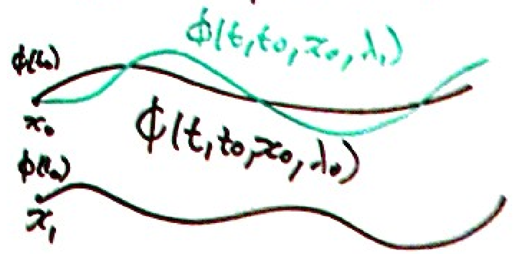
If the ODE is given by (1.1) and the pair $(t_0, x_0) \in J \times \Omega$ then is called an initial value problem.

The solution to the initial value problem is the solution of the ODE, ϕ such that

$\phi(t_0) = x_0$

If (1.1) is a family of ODE then we can consider families of solutions when we denote the solutions depending on different parameter values.

For instance $t \longmapsto \phi(t, t_0, x_0, \lambda)$



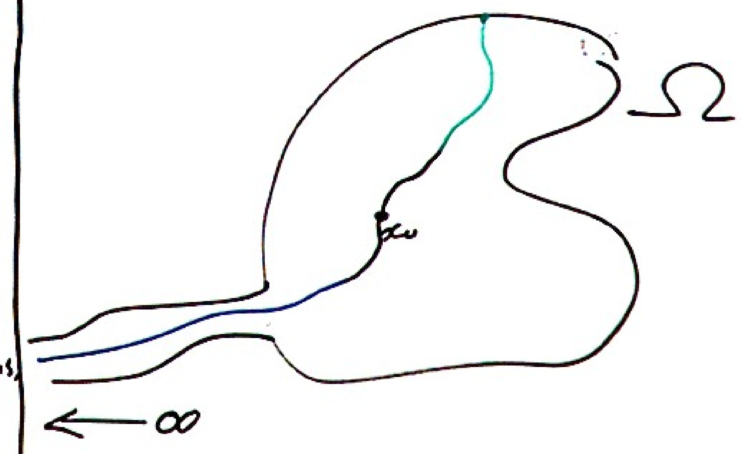
Features of ODE theory

Existence and uniqueness

The main features in the theory of ODE

- Existence
- Uniqueness
- Extensibility
- Continuity with respect to parameters

solutions to initial value problems



We have the following foundational results:

Every initial value problem (IVP) has a unique solution that is smooth with respect to initial conditions and parameters.

Moreover, the soln to the (IVP) can be extended in time until:

- It reaches the boundary of the domain of definition.

or

- The solution blows up to infinity.

3 foundational results

Existence and uniqueness

3 theorems (foundational)

Thm (1.2) (Existence and uniqueness)

$J \subseteq \mathbb{R}$, $\Omega \subseteq \mathbb{R}^n$, $\Lambda \subseteq \mathbb{R}^k$ open sets

$f : J \times \Omega \times \Lambda \to \mathbb{R}^n$ which is smooth,

$(t_0, x_0, \lambda_0) \in J \times \Omega \times \Lambda$

$\Rightarrow$ There exist open sets $J_0 \subseteq J$, $\Omega_0 \subseteq \Omega$, $\Lambda_0 \subseteq \Lambda$ with $(t_0, x_0, \lambda_0) \in J_0 \times \Omega_0 \times \Lambda_0$ and a function $\phi : J_0 \times J_0 \times \Omega_0 \times \Lambda_0 \to \mathbb{R}^n$ given by $(t, s, x, \lambda) \longmapsto \phi(t, s, x, \lambda)$ and is the only solution defined in J_0 of the IVP given by the ODE (1.1) and the initial condition $x(t_0) = x_0$.

Notation $f \in C^k$, f defined in a open set and is continuous and its partial derivatives are continuous in the open set.

Thm (1.3) (Cont. dependence)

(1.1) satisfies the hypotheses of thm (1.2)
$\Rightarrow$ the soln $\phi : J_0 \times J_0 \times \Omega_0 \times \Lambda_0 \to \mathbb{R}^n$ of (1.1) is a smooth function.
Moreover, if $f \in C^k$ with $k = 1, 2, \dots, \infty$ (resp. analytic), then $\phi \in C^k$ (resp. analytic).

Thm (1.4) (Extensibility)

(1.1) satisfies the hypotheses of Thm (1.2) and the maximal interval of existence of the soln $t \mapsto \phi(t)$ is given by (α, β) with $-\infty \le \alpha < \beta < \infty$ then $|\phi(t)|$ either approximates ∞ or $\phi(t)$ approximates a point of the boundary of Ω when $t \to \beta$.

When there is a T such that $\lim_{t \to T} |\phi(t)| = \infty$ we say that the soln blows up in finite time.

Blow up example

Existence and uniqueness

Ex $\dot{x} = x^2$, $x \in \mathbb{R}$, $t_0 = 0$, $\phi(0) = x_0$

$\phi(t) = 0$ ✓

$$\frac{d\phi(t)}{dt} = (\phi(t))^2 \quad \text{if } \phi(t) \neq 0$$

$$\frac{d\phi(t)}{dt} \Big/ \phi^2(t) = 1 \Rightarrow \frac{d}{dt}\left(-\frac{1}{\phi(t)}\right) = 1$$

$$\int_{t_0=0}^{t} \frac{d}{ds}\left(\frac{-1}{\phi(s)}\right) ds = t \Rightarrow +\frac{1}{\phi(t)} = +\frac{1}{x_0} - \frac{t x_0}{x_0}$$

$$\phi(t) = \frac{x_0}{1 - t x_0}$$

If $x_0 > 0$

$t \in (-\infty, 1/x_0) = J_0$

$1 - t x_0 = 0$

$t x_0 = 1$

$t = 1/x_0$

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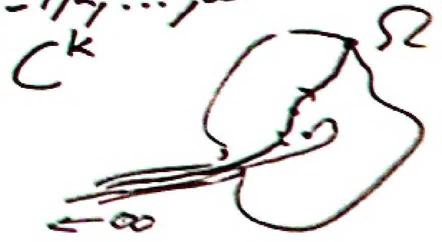

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