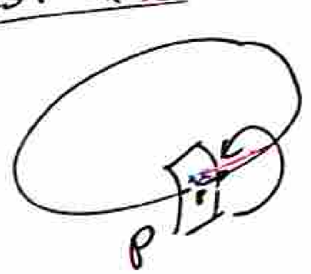
Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

Renato Calleja, 18 de abril de 2024

Last class



$\text{eig}\{DP(p)\} \cup \{1\} =$ Floquet multipliers of $\dot{W} = Df(u(t,p))W$

Thm Γ is a periodic orbit of $\dot{u} = f(u)$ and P is the corresponding Poincaré map corresponding to the section Σ s.t. $p \in \Sigma \cap \Gamma$. If the eigenvalues of $DP(p)$ are inside the unit circle, then Γ is asymptotically stable.

Asymptotic stability

There exists an ε s.t. $\forall x_0$ for which $\text{dist}(x_0, \Gamma) = \inf_{q \in \Gamma} |x_0 - q| < \varepsilon$. Γ is asymptotically stable if $\lim_{t \to \infty} \text{dist}(u(t, x_0), \Gamma) = 0$.

Lemma A an $n \times n$ matrix (possibly complex) is similar to an upper triangular matrix with entries equal to the eigenvalues of A. (weaker Jordan Canonical Form).

Def Spectral radius of A

$$\rho(A) = \sup\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$$

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p

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Proposition A is an $n \times n$ matrix. If $\varepsilon > 0$, then there exists a norm on $\mathbb{C}^n$ such that $\|A\|_\varepsilon < \rho(A) + \varepsilon$. If A is real, then the restriction to $\mathbb{R}^n$ satisfies the same property.

Pf By the lemma, $\exists\, Q$ s.t. $QAQ^{-1} = D + N$

D - diagonal, N - upper diagonal. Choose $\mu > 0$ and define $S = \text{diag}[1, \mu, \mu^2, \dots, \mu^{n-1}]$. $\rightarrow (SQA = (D + SNS^{-1})SQ)$

$$\Rightarrow S(D+N)S^{-1} = D + SNS^{-1} = D + O(\mu)$$

Define $|v|_\mu^2 = |SQv|^2 = \langle SQv, SQv \rangle$

$$|Av|_\mu^2 = \langle SQAv, SQAv \rangle = \langle (D + SNS^{-1})SQv, (D + SNS^{-1})SQv \rangle$$

$$= \langle DSQv, DSQv \rangle + \langle DSQv, SNS^{-1}SQv \rangle + \langle SNS^{-1}SQv, DSQv \rangle + \langle SNS^{-1}SQv, SNS^{-1}SQv \rangle$$

$$\leq \rho^2(A)\, |v|_\mu^2 + 2\rho(A)\, |SQv|\,|SNS^{-1}SQv| + |SNS^{-1}SQv|^2$$

$$\leq (\rho(A) + O(\mu))^2\, |v|_\mu^2 \qquad \|A\|_\mu \leq \rho(A) + O(\mu)$$

then take $\mu = \varepsilon \Rightarrow \|A\|_\varepsilon \leq \rho(A) + \varepsilon$ //

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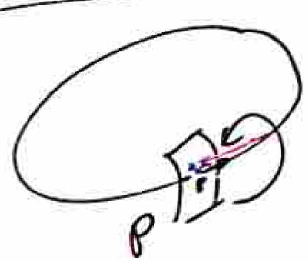

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Cor If all the eigenvalues of A are inside the unit circle, then there is an adapted norm such that $|Av|_a \le \lambda |v|_a \ \forall v \in \mathbb{R}^n$, $\lambda \in (0,1)$. In particular, A is a contraction with respect to this norm and $\lim_{n \to \infty} |A^n v|_a = 0$. (In finite dimensions all the norms are equivalent.)

$\hookrightarrow |A^k v| \le C \lambda^k |v|$

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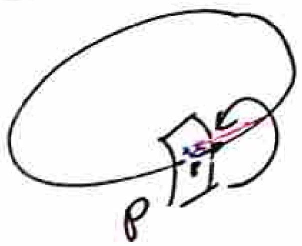

eig $\{D\mathcal{P}(p)\} \cup \{1\}$ = Floquet multipliers of $\dot{W} = Df(u(t,p))W$

Thm Γ is a periodic orbit of $\dot{u} = f(u)$ and $\mathcal{P}$ is the corresponding Poincaré map corresponding to the section Σ s.t. $p \in \Sigma \cap \Gamma$. If the eigenvalues of $D\mathcal{P}(p)$ are inside the unit circle, then Γ is asymptotically stable.

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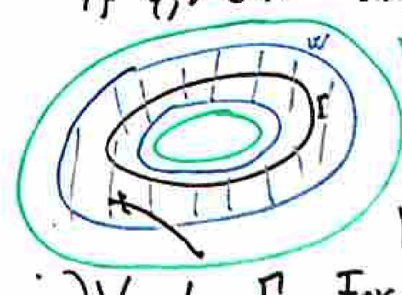
Lemma Suppose that $V \subset \mathbb{R}^n$ open set containing Γ whose compact closure is in the domain of f. If $t^* \geq 0$, then there exists $W \subseteq V$ open set containing Γ such that for each $\xi \in W$, the solution $u(t,\xi)$ is defined and stays in V for all times $t \in [0, t^*]$. Moreover, there is an $L > 0$ s.t. if $\xi, \nu \in W$ and $t \in [0, t^*]$, $|u(t,\xi) - u(t,\nu)| \leq |\xi - \nu| e^{Lt^*}$



V $(\bar{V} \subset \operatorname{dom}(f))$

Pf The estimate comes from Grönwall's inequality and $\operatorname{Lip}(f)$.

Let $m > 0$ is the minimum positive distance from ∂V to Γ. For $q \in \Gamma$, define $W_q = \{\xi \in \mathbb{R}^n : |\xi - q| e^{Lt^*} < m\}$. Using the extension thm, any soln $u(t,\xi)$ for $\xi \in W_q$ must exist for $t \in [0, t^*]$. Finally $W = \bigcup_{q \in \Gamma} W_q \subseteq V$. //

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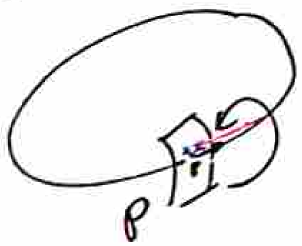

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Recall the distance between sets.

$$\operatorname{dist}(A,B) = \inf\{|a-b| : a \in A, b \in B\}$$ (It's not a metric).

Prop If $\sigma \in \Sigma$ and $\lim_{n\to\infty} P^n(\sigma) = p$ then

$$\lim_{t\to\infty} \operatorname{dist}(u(t;\sigma), \Gamma) = 0$$

Pf $\varepsilon > 0$ given, $\Sigma_0 \subset \Sigma$ open with compact closure in Σ, $p \in \Sigma_0$. The return map $T : \Sigma \to \mathbb{R}$ then take $\tau := \sup\{T(\eta) : \eta \in \Sigma_0\}$. Take V satisfing the asumptions in the previous lemma and consider W for $t^* = \tau$. Choose $\delta > 0$ suff small so that

$$\Sigma_\delta := \{\eta \in \Sigma : |\eta - p| < \delta\} \subset W \cap \Sigma_0$$

$$|\eta - p| e^{L\tau} < \min(m, \varepsilon) \quad \forall \eta \in \Sigma_\delta$$

By hypothesis, there is a $N \in \mathbb{N}$ s.t. $P^n(\sigma) \in \Sigma_\delta$, $n \geq N$.

For any $t \geq \sum_{j=0}^{N-1} T(P^j(\sigma))$ there is an $s \in [0,\tau]$ and $n \geq N$ s.t.

$$t = \sum_{j=0}^{n-1} T(P^j(\sigma)) + s$$

$$\operatorname{dist}(u(t;\sigma), \Gamma) = \inf_{q \in \Gamma} |u(t;\sigma) - q| \leq \left|u\left(s + \sum_{j=0}^{n-1} T(P^j(\sigma)); \sigma\right) - u(s;p)\right|$$

$$= \left|u\left(s; u\left(\sum_{j=0}^{n-1} T(P^j(\sigma)); \sigma\right)\right) - u(s;p)\right| = |u(s; P^n(\sigma)) - u(s;p)| < \varepsilon, \text{ n large enough.}$$

Thm Γ p.o. of $\dot{u}=f(u)$, P - Poincaré map $\to \Sigma$ s.t. $p\in\Sigma\cap\Gamma$. If eig $(DP(p))$ are in the unit circle, then Γ is asympt. stable.

Pf Suppose V neighborhood of Γ, we show that $\exists\ \mathcal{U}\subseteq V$ nghbd of Γ s.t. $u(t,\xi)\subset V\ \forall t\geq 0$ and $\text{dist}(u(t,\xi),\Gamma)\xrightarrow[t\to\infty]{} 0$. (Without loss of generality we choose V to have compact closure inside the domain of f)

By the lemma $\exists W$ open set $\Gamma\subseteq W\subseteq\bar{V}$ s.t. $\xi\subset W$, $u(t,\xi)\in V$ whenever $0\leq t\leq 2T^*$ (period of Γ)

We can suppose that $p=0$.
By the hypotheses, the spectrum of $DP(0)$ is stable and there is a norm so that $\|P(0)\|<\lambda<1$
By continuity of DP, T and the norm there is an open ball $\Sigma_0\subset\Sigma\cap W$ that contains 0 and s.t. $\sigma\in\Sigma_0$, $T(\sigma)\leq 2T^*$, $\|DP(\sigma)\|<\lambda$

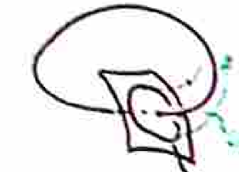
If $\sigma\in\Sigma_0$ then $P(\sigma)\subset\Sigma_0$

Obs The mean value theorem says that

$$|P(\sigma)-P(0)|\leq\|DP(\sigma^*)\|\,|\sigma-0|$$
$$|P(\sigma)|\leq\lambda|\sigma|$$

Therefore $u(t,\sigma)$ exists for all $t\geq 0$.
Also $u(T(\sigma),\sigma)=P(\sigma)\in\Sigma_0$
and in fact $u(t,u(T(\sigma);\sigma))=u(t+T(\sigma);\sigma)$
- Notice that we can repeat the argument indefinitely since we never leave Σ_0.

· Let's define $\mathcal{U}=\{u(t;\sigma):\sigma\in\Sigma_0,\ t\geq 0\}$



Σ_0

Notice that $\Gamma\subseteq\mathcal{U}$ and any flow that is in $\mathcal{U}$ is defined for all time.
We need to verify that the set $\mathcal{U}$ is open, we will do it by checking that trajectories in $\mathcal{U}$ do not separate. ...