Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

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One more example (center manifold)

- The center manifold is less regular than the v.f.
- The center manifold is not unique.

Example of non-uniqueness around the origin.

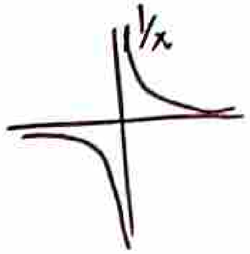
$$\dot x = x^2 \qquad \tilde x = \begin{pmatrix}0\\0\end{pmatrix} \qquad \begin{pmatrix}\dot x\\ \dot y\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}x^2\\ 0\end{pmatrix}$$
$$\dot y = -y$$

$y = h(x)$

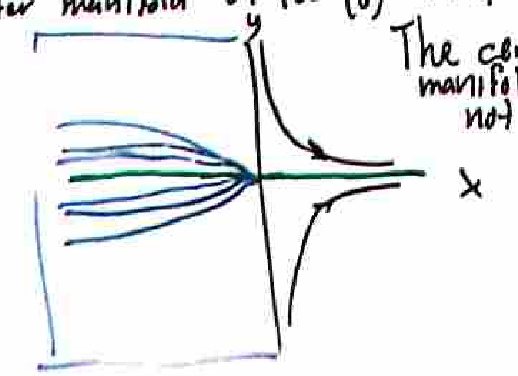
the flow $\varphi_t\binom{x_0}{y_0} = (x(t), y(t)) = \left(\frac{x_0}{1-x_0 t},\ y_0 e^{-t}\right)$

When $x_0 < 0$ there is a problem with the center manifold.

$y = A e^{1/x}$ is a center manifold of the $\binom{0}{0}$ $\forall A$.



$\xrightarrow{Ae^{(\cdot)}}$



The center manifold is not unique.

How to check that $y = Ae^{1/x}$ is indeed a center manifold.

$$y(t) = A e^{1/x(t)}$$
$$\dot y(t) = A\left(-\frac{\dot x(t)}{x^2(t)}\right) e^{1/x(t)} = y(t)(-1)$$

It satisfies the diff eq.

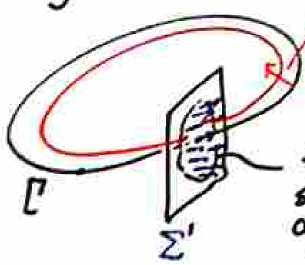
$\boxed{\text{Check that } y = Ae^{1/x} \text{ is tangent to the center space. (Volunteers?)}}$

Poicaré maps & dynamics close to periodic orbits

• Topic of HW #5 will be posted later today.

The dynamics close to a periodic orbit are studied by means of a Poincaré map.

Directions that are normal to the p.o. are more relevant.



the section is also transversal to the flow

$f(\sigma)$ transversal to Σ

$\varphi_t(\sigma)$

there is a t such that $\varphi_t(\sigma) \in \Sigma$

$P(\sigma) = \varphi_t(\sigma)$

Let $\dot{u} = f(u)$, $u \in \mathbb{R}^n$ has a periodic orbit named $\Gamma \subseteq \mathbb{R}^n$

Let $u(t,\xi)$ be the solution (flow) with $u(0,\xi) = \xi$.

If $p \in \Gamma$ and $\Sigma' \subset \mathbb{R}^n$ is a section (co-dimension 1 submanifold of $\mathbb{R}^n$) that is transversal to Γ at p.

Transversal means that $f(p) \in T_p \Sigma'$

the vector $f(p)$ is not in the tangent space of Σ' at p.

Then by the IFT we know that there is an open set Σ of Σ' where the v.f. is transversal at every point of Σ. $\Sigma \subseteq \Sigma'$

$\exists\, T : \Sigma \to \mathbb{R}$ $\forall \sigma \in \Sigma$, $u(T(\sigma), \sigma) \in \Sigma'$

The time that needs to pass so that the flow is back at Σ'

The number $T(\sigma)$ is called the <u>first</u> return map of the point σ.

Poicaré maps & dynamics close to periodic orbits

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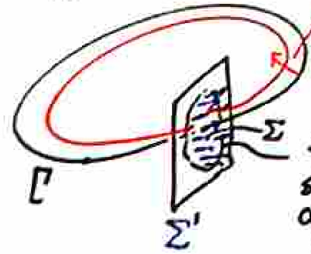

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$f(\sigma)$ transversal to Σ

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there is a t such that $\varphi_t(\sigma) \in \Sigma$

$P(\sigma) = \varphi_t(\sigma)$

The <u>Poinaré map</u> is defined by

$P : \Sigma \to \Sigma'$, $P(\sigma) = u(T(\sigma), \sigma) \in \Sigma$

<u>Note</u>. that P depends on the choice of Σ.

- the Poincaré map describes the dynamics on the Poincaré section Σ.

The function could be complicated since it is defined though the flow $u(t, \xi)$.

Something simple about P is that if $p \in \Gamma$ then $P(p) = p$. $T(p) = T^*$ with T^* the period of Γ

$$P(p) = u(T(p), p) = u(T^*, p) = u(0, p) = p$$

$\exists\, T : \Sigma \to \mathbb{R}$ $\forall \sigma \in \Sigma$, $u(T(\sigma), \sigma) \in \Sigma'$

The time that needs to pass so that the flow is back at Σ'

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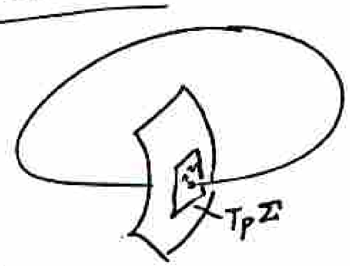
Poicaré maps & dynamics close to periodic orbits

If $v \in \mathbb{R}^n$ that is a tangent vector to Σ at p, $v \in T_p \Sigma$



then the derivative of P at p is related to the derivative of the flow in the direction v and the derivative of the v.f. $(\dot u = f(u)$

$$DP(p)v = D_\sigma u(T(\sigma),\sigma)\Big|_{\sigma=p}\cdot v$$

$$= \frac{du}{dt}(\overset{T^*}{T(p)},p)\cdot \frac{d}{d\sigma}T(p)\cdot v + D_\xi u(\overset{T^*}{T(p)},p)\cdot v$$

$$= f(u(T^*,p))\,\nabla T(p)\cdot v + D_\xi u(T^*,p)\cdot v$$

$$= f(p)\,\nabla T(p)\cdot v + D_\xi u(T^*,p)\cdot v$$

In general $DP(p) = f(p)\nabla T(p) + D_\xi u(T^*,p)$

Notice that although the dimension of Σ is $n-1$ we are thinking that Σ is a subset of $\mathbb{R}^n$ and we are using the associated coordinate system of $\mathbb{R}^n \Rightarrow DP(p)$ as an $n\times n$ matrix.

→ The Poinaré map is defined by

$P:\Sigma \to \Sigma'$, $P(\sigma) = u(T(\sigma),\sigma)\in\Sigma$

Note: • that P depends on the choice of Σ.

• the Poincaré map describes the dynamics on the Poincaré section Σ.

The function could be complicated since it is defined though the flow $u(t,\xi)$.
Something simple about P is that if $p\in\Gamma$ then $P(p)=p$. $T(p)=T^*$ with T^* the period of Γ

$$P(p) = u(T(p),p) = u(T^*,p) = u(0,p) = p$$

$\exists\, T:\Sigma\to\mathbb{R}$ $\forall \sigma\in\Sigma$, $u(T(\sigma),\sigma)\in\Sigma'$

The time that needs to pass so that the flow is back at Σ'

The number $T(\sigma)$ is called the first return map of the point σ.

Poicaré maps & dynamics close to periodic orbits

If $v \in \mathbb{R}^n$ that is a tangent vector to Σ at p, $v \in T_p \Sigma$

then the derivative of $\mathcal{P}$ at p is related to the derivative of the flow in the direction v and the derivative of the v.f. ($\dot{u} = f(u)$)

$$
\begin{aligned}
D\mathcal{P}(p)v &= D_\sigma u(T(\sigma),\sigma)\big|_{\sigma=p}\cdot v \\
&= \frac{du}{dt}(\overset{T^*}{T(p)},p)\cdot \frac{d}{d\sigma}T(p)\cdot v + D_\xi u(\overset{T^*}{T(p)},p)\cdot v \\
&= f(u(T^*,p))\, \nabla T(p)\cdot v + D_\xi u(T^*,p)\cdot v \\
&= f(p)\, \nabla T(p)\cdot v + D_\xi u(T^*,p)\cdot v
\end{aligned}
$$

In general $D\mathcal{P}(p) = f(p)\nabla T(p) + D_\xi u(T^*,p)$

Notice that althoug the dimension of Σ is $n-1$ we are thinking that Σ is a subset of $\mathbb{R}^n$ and we are using the associated coordinate system of $\mathbb{R}^n \Rightarrow D\mathcal{P}(p)$ as an $n \times n$ matrix.

Also note that besides p that is a fixed point of $\mathcal{P}$ ($\mathcal{P}(p)=p$) there could also be fixed points for $\mathcal{P}^k(q)=q$ that are not fixed points of $\mathcal{P}(q)\neq q$, for $k\in\mathbb{N}$.

We say that q is a periodic point of $\mathcal{P}$ with period $= 2$. $\mathcal{P}^2(q)=q$, $\mathcal{P}(q)\neq q$.

In terms of the flow, $u(t,q)$ is a periodic orbit that winds twice around the Poincaré section.

Analogously a point ξ_k s.t. $\mathcal{P}^k(\xi_k)=\xi_k$ but $\mathcal{P}^j(\xi_k)\neq \xi_k$ $j=1,\dots,k-1$ belongs to a periodic orbit that wind around the Poincaré section k times.

→ Going back to the derivative $D\mathcal{P}$. We intuitively know that the eigenvalues of this matrix are related to the stability of the periodic orbit.

→ By noticing that $\dot{w} = Df(u(t,p))w$ ($\dot{w} = A(t)w$ where $A(t+T^*)=A(t)$)

... so there should be a connection between the Floquet multipliers/exponents and the eigenvalues of $D\mathcal{P}(p)$.

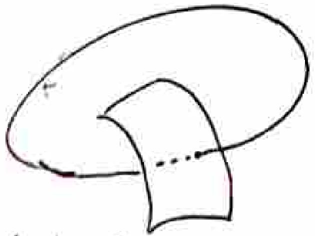
Poicaré maps & dynamics close to periodic orbits

$$DP(p)v = f(p)\nabla T(p)v + D_\xi u(T^*, p)v.$$

Prop (Chicone, Prop 2.122)
Γ is a per. orb. & $p \in \Gamma$ then the union of the set of eigenvalues of the derivative of the Poinaré map and the set $\{1\}$ is equal to the set of characteristic multipliers of the first variational equation along Γ.
In particular, 0 is not an eigenvalue of $DP(p)$.

Pf From this expression, $v \in \mathbb{R}^n$

Write $v = (f(p), \sigma_1, \ldots, \sigma_{n-1}) \in \mathbb{R}^n$



If v is pointing in the direction of Σ, then the first component of v is going to be 0.

→ We can identify the derivative of the Poinaré map with a matrix B $((n-1)\times(n-1))$ and the derivative of the return map with A $(1 \times (n-1))$

→ In the direction of $f(p)$ we only have a component in the first entry.

Remember that: $D_\xi u(t,p)$ is the principal fundamental matrix solution of the linearized equation around the solution $u(t,p)=\Gamma$. By Floquet's theorem, we write

$$D_\xi u(t,p) = P(t)e^{t\bar{B}}, \text{ with } P(0)=I$$

an in this basis,

$$n-1\left\{\begin{pmatrix} 1 & A \\ \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \underbrace{B}_{n-1} \end{pmatrix}\right. = D_\xi u(T^*,p) = P(T^*)e^{T^*\bar{B}} = e^{T^*\bar{B}}$$

Since the eigenvalues $e^{T^*\bar{B}}$ are the Floquet's multipliers then we obtain the conclusion of the theorem.

→ this gives an eigenvalue $=1$, so one Floquet multiplier is $=1$.