Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

Renato Calleja, 16 de mayo de 2024

Thm

Ω is compact, non-empty and is the ω-limit of a flow. If Ω does not contain a fixed point, then it is a periodic orbit.

Pf Suppose that $\omega(p)$ is compact, nonempty and doesn't contain a fixed point.

$q \in \omega(p)$ $\quad q \to \omega(q)$

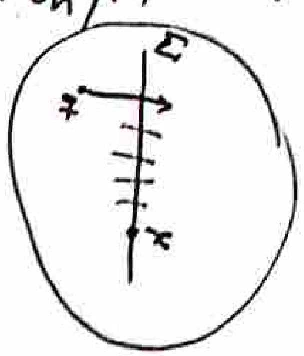
$\omega(q) \subset \omega(p)$

take $x \in \omega(q)$

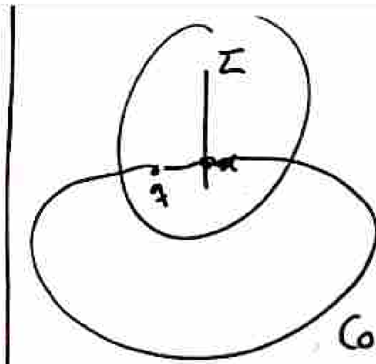
$\varphi(t_n, q) \to x$, $\varphi(s_n, p) \to q$

$$\varphi(s_n + t_n, p) = \varphi(t_n, \varphi(s_n, p)) \to x$$

$x \in \omega(p)$



All these points belong to $\omega(p)$. So by the lemma all the points are equal



q belongs to a closed orbit. Γ

$q \in \Gamma \subset \omega(p)$ by invariance.

We only need to check that $\omega(p) \subset \Gamma$

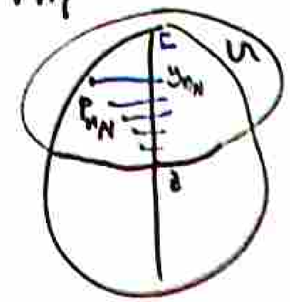
Let's assume that $\omega(p) \setminus \Gamma \neq \emptyset$

Consider $A_r = \bigcup_{q \in \Gamma} B_r(q)$

Take a sequence $\{p_n\}_{n=1}^{\infty} \subseteq \omega(p) \setminus \Gamma$ such that

$p_n \in A_{r_n}$, with $r_n \to 0$ so $\operatorname{dist}(p_n, \Gamma) \xrightarrow[n \to \infty]{} 0$

Let p_{n_i} be a convergent sub-sequence $p_{n_i} \xrightarrow[i \to \infty]{} z \in \Gamma$



We can rectify in U

$\exists N \in \mathbb{N}$ s.t. $p_{n_i} \in U$, $i > N$

$y_{n_N} \subset \omega(p)$. This is a contradiction

So $\omega(p) \setminus \Gamma$ has to be empty. //

$\omega(p) = \Gamma$

Bifurcation Theory

- I use the book of Kuznetsov(?)
- Golubitsky (?)

Consider the diff. eq.

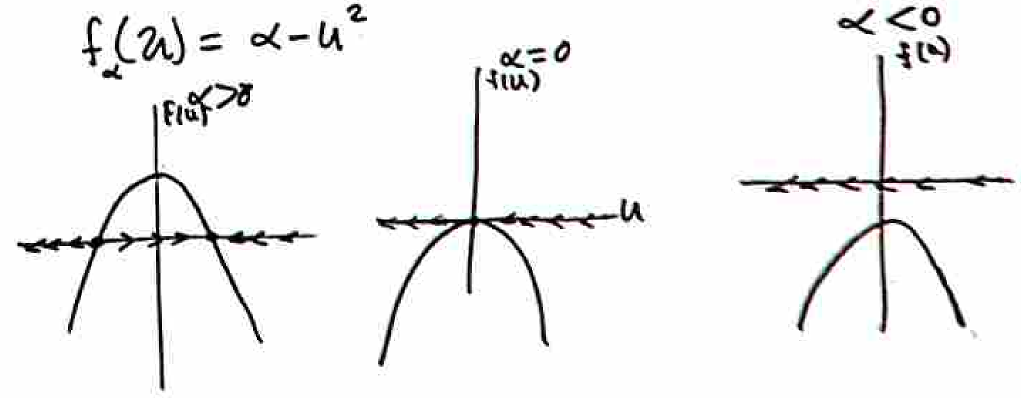
$$\dot u = f(u,\alpha), \quad u\in\mathbb{R}^n, \quad \alpha\in\mathbb{R}^m. \qquad (1)$$

We say that (1) under goes a bifuration if there is an α_0 and arbitrarily close values to α_0 for which the behaviour of (1) is qualitatively different.

First example

$\dot u = \alpha - u^2$, $u\in\mathbb{R}$, $\alpha\in\mathbb{R}$.

$f_\alpha(u) = \alpha - u^2$



- Fixed points.

2 1 no

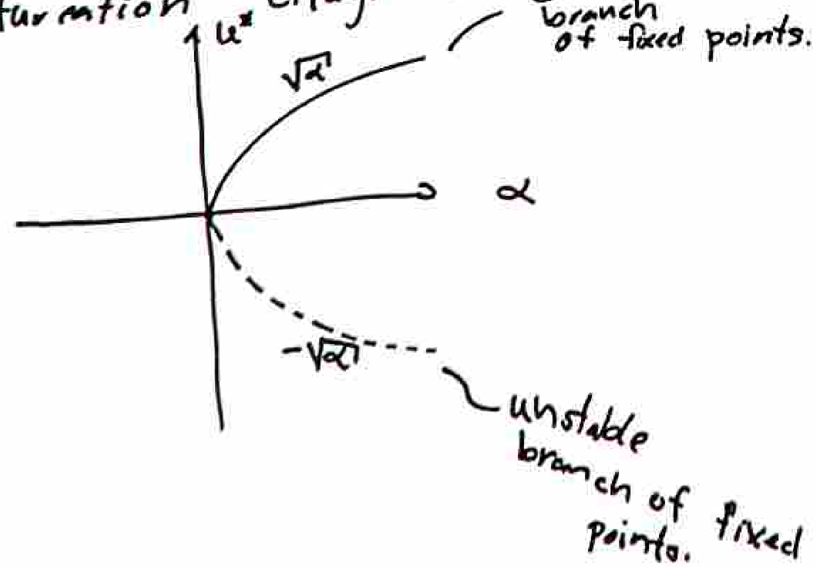
They are qualitatively different.

Bifurcation Theory

First example (Saddle-node bifurcation)

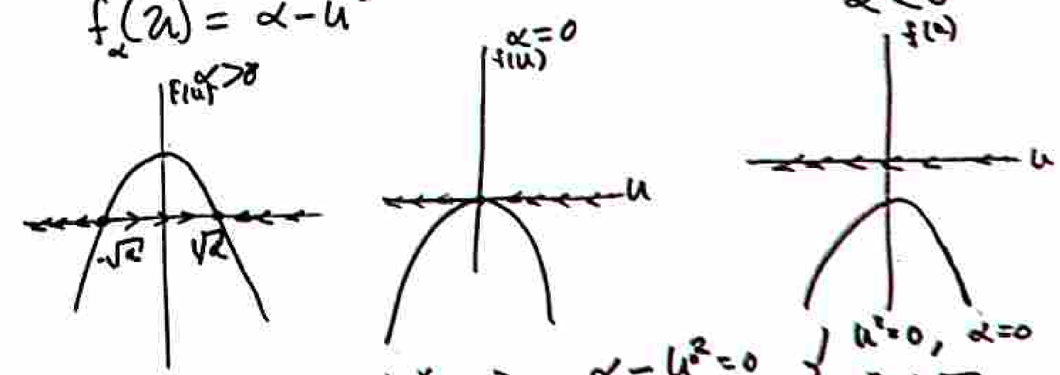
$\dot{u} = \alpha - u^2$, $u \in \mathbb{R}$, $\alpha \in \mathbb{R}$ (The normal form of the s-n bifn)

A bifurcation diagram



$\dot{u} = \alpha - u^2$, $u \in \mathbb{R}$, $\alpha \in \mathbb{R}$.

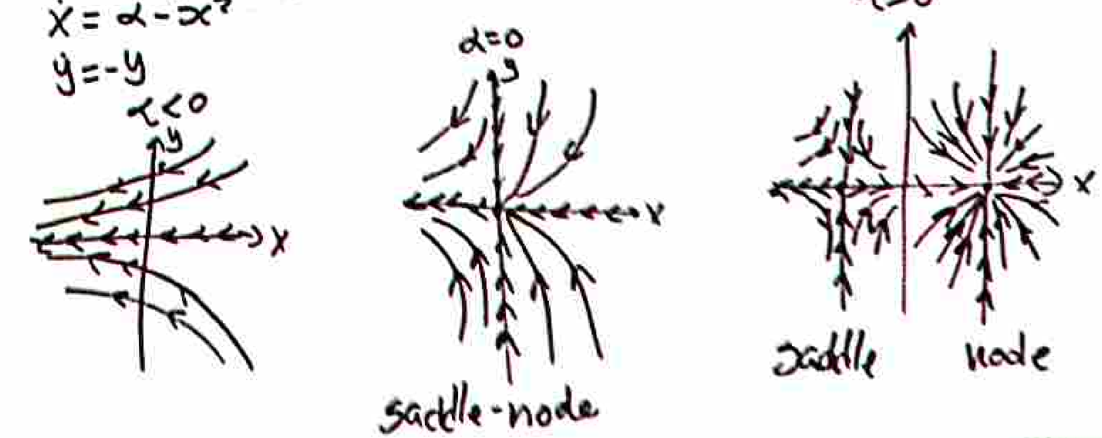
$f_\alpha(u) = \alpha - u^2$



Fixed points. $u^* \Rightarrow \alpha - u^{*2} = 0$ $\begin{cases} u^* = 0, & \alpha = 0 \\ u^* = \pm\sqrt{\alpha} \end{cases}$

$\dot{x} = \alpha - x^2$

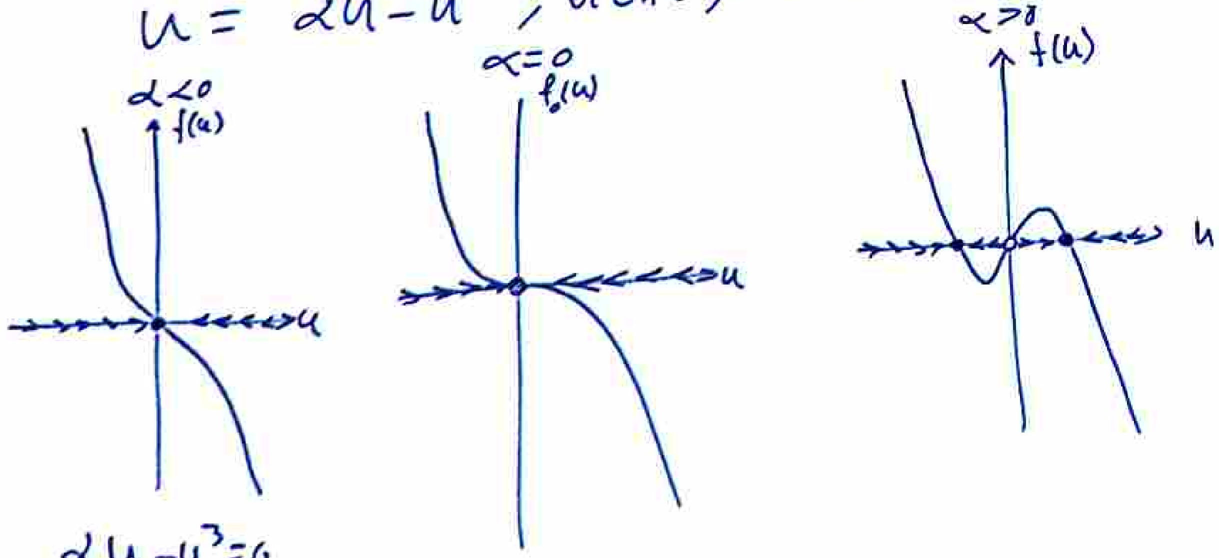
$\dot{y} = -y$



Bifurcation Theory

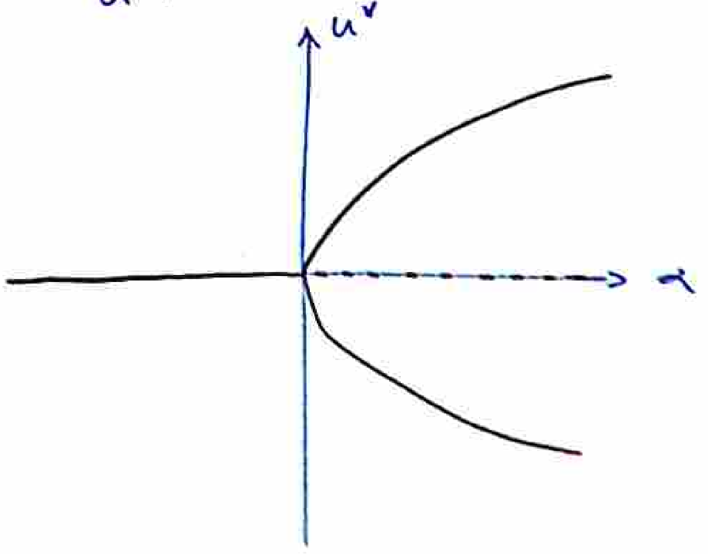
2^{nd} example (Pitchfork bifurcation)

$\dot{u} = \alpha u - u^3$, $u \in \mathbb{R}$, $\alpha \in \mathbb{R}$. $f_\alpha(u) = \alpha u - u^3$



$\alpha u - u^3 = 0$
$u(\alpha - u^2) = 0$

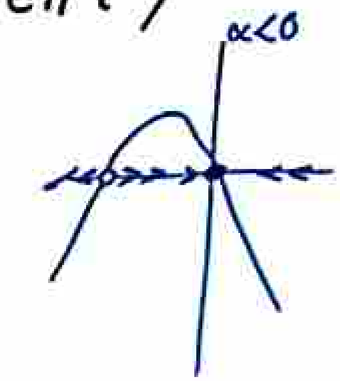
Bifurcation diagram
α vs u^*



Bifurcation Theory

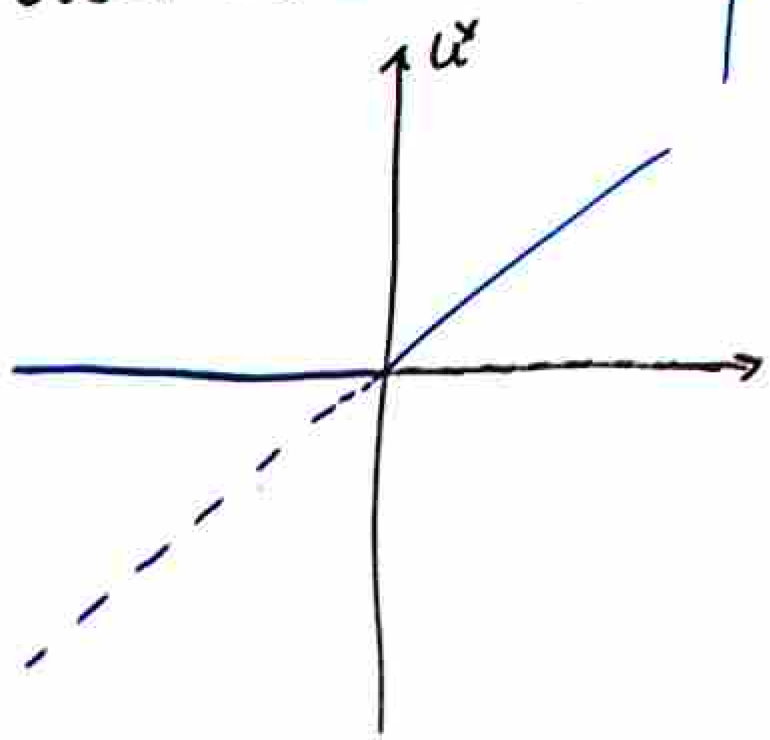
3^nd example (Transcritical bifurcation)

$$\dot{u} = \alpha u - u^2, \quad u \in \mathbb{R}, \quad \alpha \in \mathbb{R}$$



Fixed points

$$u^*(\alpha - u^*) = 0$$



The branches exchange stability.

Bifurcation Theory

4th example (Andronov-Hopf bifurcation)

$$\dot{x} = \alpha x - y + s\,x(x^2+y^2)$$
$$\dot{y} = x + \alpha y + s\,y(x^2+y^2)$$

$s = \pm 1$

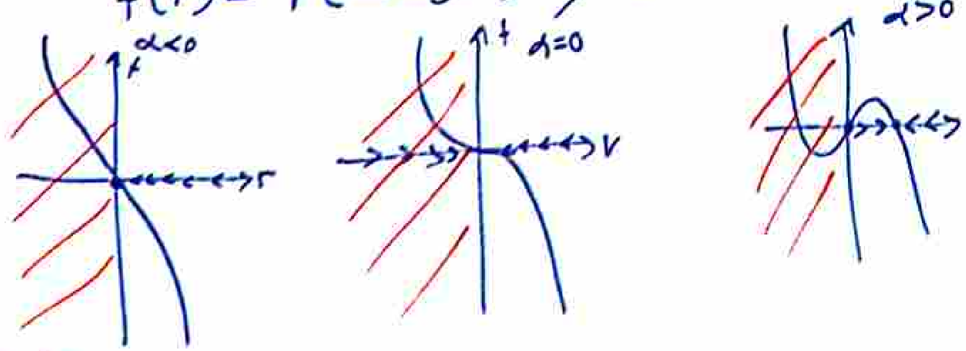
$(x,y)^T \in \mathbb{R}^2, \ \alpha \in \mathbb{R}$

In polar coordinates

$r^2 = x^2+y^2, \quad \tan(\theta) = y/x \Rightarrow \dot{\theta} = \dfrac{\dot{y}x - \dot{x}y}{r^2}$

$\dot{r} = r(\alpha - s r^2), \quad \dot{\theta} = 1$ (A-H normal form)

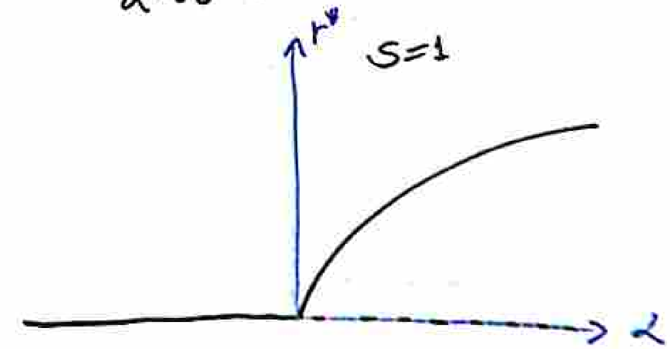
$f(r) = r(\alpha - s r^2), \quad s = 1$

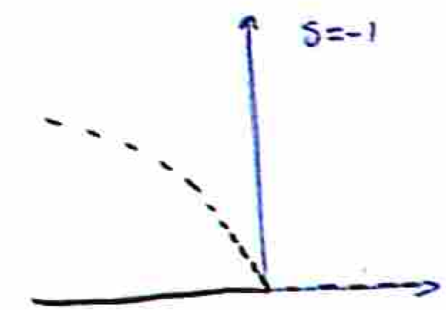


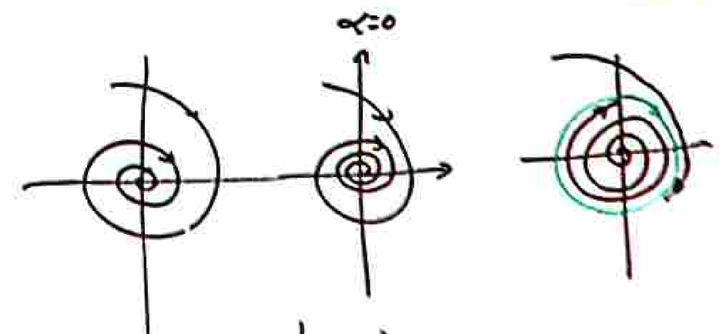
r is always positive

Bfn diagram

α vs r^*

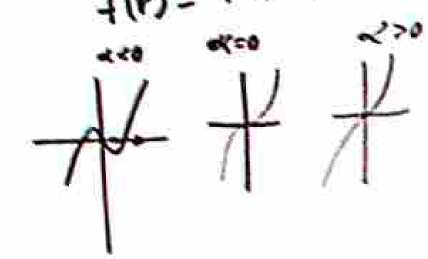






Super-critical A-H bfn

$f(r) = r(\alpha + r^2)$



Sub-critical A-H bfn.