Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

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Solution matrices

Cor

Let $\{x_i(t_0)\}_{i=1}^n$ be linearly independent, then $\{x_i(t)\}_{i=1}^n$ are linearly independent $\forall t \in J$.

$$\dot{x} = A(t)x$$

Def

·) An $n \times n$ matrix $\Psi(t)$ is a solution matrix of $\dot{x} = A(t)x$ in an interval J if each of its colums is a solution.

$$\Psi(t) = (x_1(t) \,|\, x_2(t) \,|\, \dots \,|\, x_n(t)) \in M(\mathbb{R})^{n \times n}$$

··) $\Psi(t)$ is a fundamental solution matrix if each colum is linearly independent.

···) If $\Psi(t_0) = I$ (Identity matrix) then it is the principal fundamental solution matrix.

In fact all the fundamental solutions are related by matrix multiplication.

If Ψ and Φ are fund. solns then

$$\Psi(t) = \Psi(t_0)\, \Phi^{-1}(t_0)\, \Phi(t)$$

Furthermore, for any vector $v \in \mathbb{R}^n$ $\Phi(t)v$ is a solution to $\dot{x} = A(t)x$.

· Every solution to the equation can be written in this form.

Liouville's formula

Liouville's formula

$\Phi(t)$ a matrix solution of $\dot{x} = A(t)x$ on J

Then if $t_0 \in J$, $\det \Phi(t) = \det \Phi(t_0)\, e^{\int_{t_0}^{t} tr(A(s))ds}$

Generalization $\boxed{\text{2 proofs. 1st requires A to be differentiable; 2nd standard less intuitive, requires props of det}}$

1829

Abel's identity

$y'' + p(t)y' + q(t)y = 0$

The Wronskian of two functions y_1, y_2

$$W(y_1, y_2)(t) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0)\cdot \exp\left(-\int_{t_0}^{t} p(s)ds\right)$$

$\forall t \in J$

$$\begin{aligned} y' &= z \\ z' &= -p(t)z + q(t)y \end{aligned} \Rightarrow \begin{pmatrix} y \\ z \end{pmatrix}' = x' = \overbrace{\begin{pmatrix} 0 & 1 \\ q(t) & -p(t) \end{pmatrix}}^{A(t)} \begin{pmatrix} y \\ z \end{pmatrix}$$

$$= A(t)x$$

First proof

Liouville's formula

$\Phi(t)$ a matrix solution of $\dot{x} = A(t)x$ on J

Then if $t_0 \in J$, $\det \Phi(t) = \det \Phi(t_0)\, e^{\int_{t_0}^{t} tr(A(s))ds}$

$\boxed{\text{2 proofs. } 1^{st} \text{ requires } A \text{ to be differentiable; } 2^{nd} \text{ standard less intuitive, requires props of det}}$

1st Proof (Kapitula & Promislow) Spectral and Dynamical stability of non-linear waves.

All fundamental matrix solutions are related by $\Psi(t) = \Psi(t_0)\, \Phi(t_0)^{-1} \Phi(t)$. So we only have to check the formula for a special fundamental matrix solution

We will show

$$\frac{d}{dt} \det \Phi(t) = tr(A(t)) \det \Phi(t)$$

·) $\det \Phi(t)$ and $tr A(t)$ are functions from J to $\mathbb{R}$

··) If $\det \Phi(t) = 0$ then we are done.

If $\det \Phi(t) \neq 0$

$$\frac{\frac{d}{dt}\det \Phi(t)}{\det \Phi(t)} = tr A(t)$$

$$\int_{t_0}^{t} \frac{d}{ds}\left(\log(\det \Phi(s))\right) ds = \int_{t_0}^{t} tr A(s)\, ds$$

$$\iff \det \Phi(t) = \det \Phi(t_0)\, e^{\int_{t_0}^{t} tr A(s) ds}.$$

We only need to proove

First proof of Liouville's Formula

Liouville's formula

$\Phi(t)$ a matrix solution of $\dot{x} = A(t)x$ on J

Then if $t_0 \in J$, $\det \Phi(t) = \det \Phi(t_0)\, e^{\int_{t_0}^{t} tr(A(s))ds}$

$\boxed{\text{2 proofs. 1st requires } A \text{ to be differentiable; 2nd standard less intuitive, requires props of det}}$

1st Proof (Kapitula & Promislow)

$$\frac{d}{dt} \det \Phi(t) = tr\, A(t) \det \Phi(t)$$

$$A \text{ diff} \Rightarrow A(t) = A(t_0) + \int_0^1 \underbrace{\frac{d}{ds} A(st + (1-s)t_0)}_{(A(t)-A(t_0))} ds$$

$$= A(t_0) + (t-t_0)\int_0^1 A'(st + (1-s)t_0)\, ds$$

$$= A(t_0) + \mathcal{O}(|t-t_0|)$$

Now assume that $A(t_0)$ has a complete basis of eigenvectors $\{v_1, \dots, v_n\}$ then λ_j $j = 1, \dots, n$ eigenvalues.

Suppose that $X_j(t)$ is the soln of $\dot{x} = A(t)x$ with $X_j(t_0) = v_j$, $\dot{x} = A(t_0)x + (A(t) - A(t_0))x$

We can write the soln.

$$\boxed{X_j(t) = e^{\lambda_j(t-t_0)} v_j + y_j(t)} \quad \text{(Superposition)}$$

$$X_j(t_0) = v_j + \cancel{y_j(t_0)}^{\,0}$$

$$y_j'(t_0) = 0 \quad \text{since is satisfies} \quad y_j'(t_0) = (A(t_0) - A(t_0))\, y_j(t_0)$$

$$y_j(t) = \cancel{y_j(t_0)} + \cancel{y_j'(t_0)}(t-t_0) + \mathcal{O}(|t-t_0|^2)$$

$$= \mathcal{O}(|t-t_0|^2) \Rightarrow X_j(t) = e^{\lambda_j(t-t_0)} v_j + \mathcal{O}(|t-t_0|^2)$$

$$\det \Phi(t) = \det[X_1(t) | X_2(t) | \dots | X_n(t)]$$

$$\frac{d}{dt} \det \Phi(t) = \sum_{j=1}^{n} \det[X_1(t) | X_2(t) | \dots | X_j'(t) | \dots | X_n(t)]$$

$$= \sum_{j=1}^{n} \det[X_1(t) | X_2(t) | \dots | \lambda_j e^{\lambda_j(t-t_0)} v + y_j'(t) | \dots | X_n(t)]$$

End of the first proof of Liouville's formula

Liouville's formula

$\Phi(t)$ a matrix solution of $\dot{x} = A(t)x$ on J

Then if $t_0 \in J$, $\det \Phi(t) = \det \Phi(t_0)\, e^{\int_{t_0}^{t} tr(A(s))ds}$

$\boxed{\text{2 proofs. } 1^{st} \text{ requires } A \text{ to be differentiable; } 2^{nd} \text{ standard less intuitive, requires props of det}}$

1st Proof (Kapitula & Promislow)

$$\lambda_j e^{\lambda_j(t-t_0)} + y_j'(t) = \lambda_j X_j(t) + \mathcal{O}(|t-t_0|)$$

$$\Rightarrow \frac{d}{dt}\det \Phi(t) = \sum_{j=1}^{n} \det\left[X_1(t)|\dots|\lambda_j X_j(t)|\dots|X_n(t)\right] + \mathcal{O}(|t-t_0|)$$

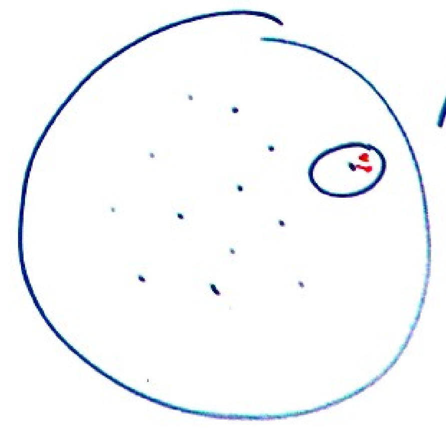
$$= \sum_{j=1}^{n}\lambda_j \underbrace{\det\left[X_1(t)| \dots |X_n(t)\right]}_{} + \mathcal{O}(|t-t_0|)$$

$$= tr(A(t_0)) \det \Phi(t) \rightarrow \text{true } \forall t_0 \in J$$

Make $t_0 = t$ $\quad \frac{d}{dt}\det\Phi(t) = tr A(t) \det \Phi(t)$

What happens if $A(t_0)$ doesn't have $\{v_j\}_{j=1}^{n}$ and λ_j, $j=1,\dots,n$

In HW 2 you will explore how to perturb matrices.



$M^{n\times n}$

You have the formula for the red matrix (that is ε close to the bad matrix)

$\frac{d}{dt}\det\Phi_\varepsilon(t) = tr A_\varepsilon(t) \det \Phi_\varepsilon(t)$

then take the limit as $\varepsilon \to 0$ and the formula still holds.

With A differentiable.

Second proof of Liouville's formula

Liouville's formula

$\Phi(t)$ a matrix solution of $\dot{x} = A(t)x$ on J

Then if $t_0 \in J$, $\det \Phi(t) = \det \Phi(t_0)\, e^{\int_{t_0}^{t} tr(A(s))ds}$

2 proofs. 1st requires A to be differentiable
2nd standard less intuitive, requires props of det

2nd Proof (Standard)

Since $\Phi(t)$ is a solution to $\dot{\Phi}(t) = A(t)\Phi(t)$

$\Phi(t)$ is differentiable, so

$$\lim_{h\to 0} \frac{1}{h}\left[\Phi(t+h) - (I + hA(t))\Phi(t)\right] = 0$$

which implies that

$$\Phi(t+h) = (I + hA(t))\Phi(t) + o(h)$$

$o(h) \sim h^{3/2}$

$$\lim_{h\to 0} \frac{o(h)}{h} = 0$$

We will prove that

$$\Phi(t+h) = (I + hA(t))\Phi(t) + o(h) = (1 + h\,tr A(t))\det\Phi(t) + o(h)$$

This can be seen by the following two facts

Set $\alpha = (I + hA(t))\Phi(t)$

$\beta = o(h)$

$$\begin{aligned}
\det(\alpha+\beta) &= \det(\alpha) + [\det(\alpha+\beta) - \det(\alpha)] \\
&= \det(\alpha) + \int_0^1 \frac{d}{ds}\det[s(\alpha+\beta) + (1-s)\alpha]\,ds \\
&= \det(\alpha) + \int_0^1 \det'[s(\alpha+\beta) + (1-s)\alpha](\alpha+\beta-\alpha)\,ds \\
&= \det(\alpha) + \beta\int_0^1 \det'[s\beta + \alpha]\,ds \\
&= \det(\alpha) + C\,o(h)
\end{aligned}$$

$$\det\alpha = \det((I + hA(t))\Phi(t)) = \det(I + hA(t))\det(\Phi(t))$$

$$\det(I+B) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\sum_{j=1}^{\infty}\frac{(-1)^j}{j!}tr(B^j)\right)^k = 1 + tr B + \mathcal{O}(tr(B^2))$$

$$\det(I + hA(t)) = 1 + h\,tr(A(t)) + o(h^2)$$

$$\det(\Phi(t+h)) = [1 + h\,tr(A(t)) + \mathcal{O}(h^2)]\det(\Phi(t))$$

$$\Rightarrow \frac{d}{dt}\Phi(t) = tr(A(t))\det(\Phi(t))$$

Autonomous linear case

Now the autonomous case

$$\dot{x} = A(t)x$$

$$\rightarrow \dot{x} = \bar{A}x \quad , A \in M^{n\times n}(\mathbb{R})$$

In the real case, we can write down the solution as a semi group e^{tA} related the exponential matrix.

Please review e^{A} exponential matrix.

Ejercicio 4 Tarea 1

$$f(x,\lambda) = |x| + |\lambda|$$

$$\boxed{\begin{array}{l}\dot{x} = f(x,\lambda)\\ \dot{\lambda} = 1\end{array}}$$

$$u = \begin{pmatrix} x \\ \lambda \end{pmatrix}$$

$$\dot{u} = F(u)$$

$$F(u) = \begin{pmatrix} f(x,\lambda) \\ 1 \end{pmatrix}$$

$$\|F(u_1) - F(u_2)\| \leq L\,\|u_1 - u_2\|$$

u