Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

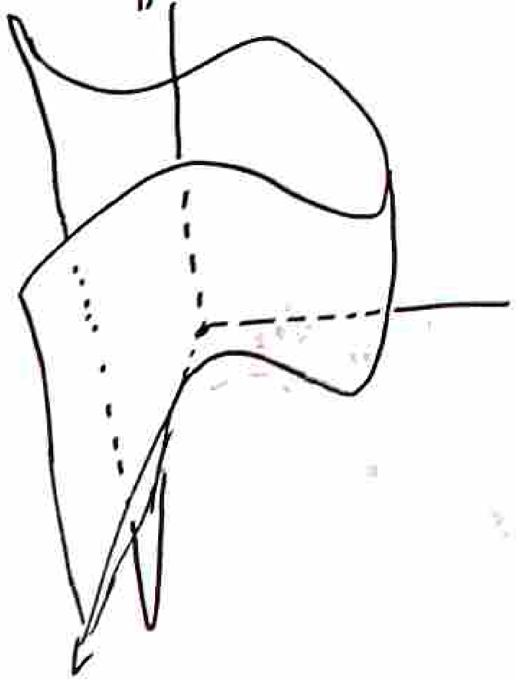
IIMAS

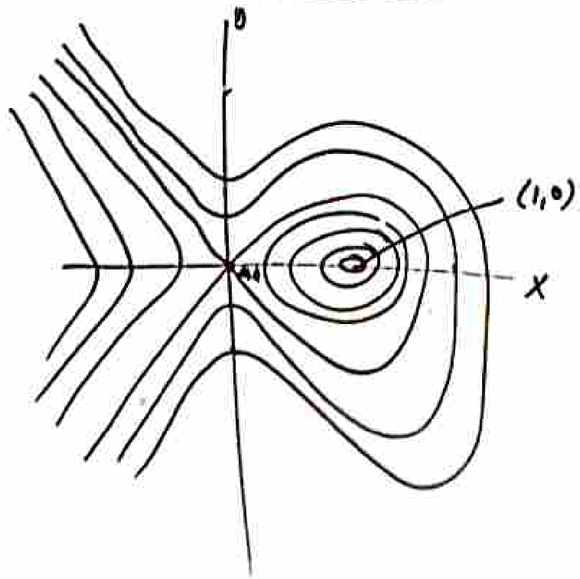
Renato Calleja, 14 de marzo de 2024

Grobman – Hartman theorem
1959   1960

p hyp. fixed point of f, φ_t restricted to a neighborhood of p the flow of $\dot{x} = f(x)$, is congugate to the flow of the linearized equation $\psi(t,y) = e^{tA}y$ in a neighborhood of the origin

$\{ A = Df(p) \}$





Ex (A hamiltonian diff. eq.)

$H: \mathbb{R}^2 \to \mathbb{R}$, $H(x,y) = 2x^3 - 3x^2 + y^2$

$$f(x,y) = \begin{pmatrix} \frac{\partial H}{\partial y}(x,y) \\ -\frac{\partial H}{\partial x}(x,y) \end{pmatrix} = \begin{pmatrix} 2y \\ -6x^2 + 6x \end{pmatrix}$$

↳ defines a Hamiltonian v.f.

$$\dot{x} = 2y \qquad \longrightarrow \varphi_t \text{ the flow of } \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f(x,y)$$
$$\dot{y} = -6x^2 + 6x \qquad \varphi_t = (x(t), y(t))$$

$$\frac{d}{dt} H(x(t), y(t)) = \frac{d}{dt}\left(2x(t)^3 - 3x(t)^2 + y(t)^2\right)$$
$$= 6x(t)^2 \cdot \dot{x}(t) - 6x(t)\cdot \dot{x}(t) + 2y(t)\cdot \dot{y}(t)$$
$$= (6x(t)^2 - 6x(t))\,2y(t) + 2y(t)(-6x^2(t) + 6x(t)) = 0$$

The level curves of H are invariant.

1) $y=0, x=0$ $\dot{x} = 0$, $\dot{y} = 0$

2) $y=0, x=1, -6x(x-1) = 0$

$$Df(x,y) = \begin{pmatrix} 0 & 2 \\ -12x+6 & 0 \end{pmatrix}$$

at 1) $Df(0,0) = \begin{pmatrix} 0 & 2 \\ 6 & 0 \end{pmatrix}$, $\lambda = \pm\sqrt{\ldots}$

at 2) $Df(1,0) = \begin{pmatrix} 0 & 2 \\ -6 & 0 \end{pmatrix}$

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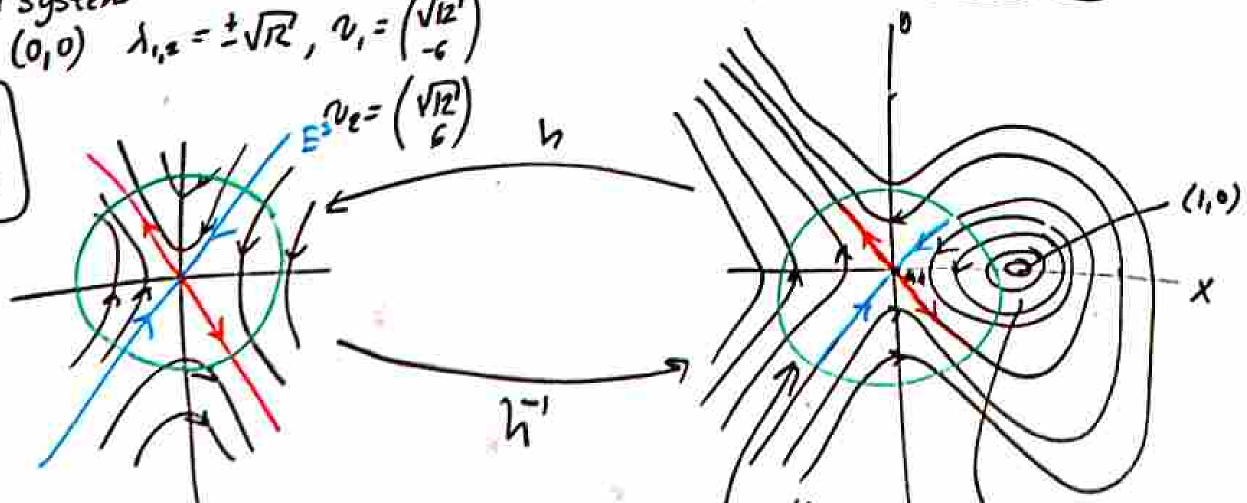
p hyp. fixed point of f, φ_t restricted to a neighborhood of p the flow of $\dot{x}=f(x)$, is congugate to the flow of the linerized equation $\psi(t,y)=e^{tA}y$ in a neighborhood of the origin $\{\, A=Df(p)$

Linerized system
At $(0,0)$ $\lambda_{1,2}=\pm\sqrt{12}$, $v_1=\begin{pmatrix}\sqrt{12}\\-6\end{pmatrix}$, $v_2=\begin{pmatrix}\sqrt{12}\\6\end{pmatrix}$

$$\boxed{\begin{aligned}\dot{x}&=2y\\ \dot{y}&=6x\end{aligned}}$$



G-H ☺ → No but we will talk about the center manifold thm.

Ex (A hamiltonian diff. eq.)

$H:\mathbb{R}^2\to\mathbb{R}$, $H(x,y)=2x^3-3x^2+y^2$

$$f(x,y)=\begin{pmatrix}\frac{\partial H}{\partial y}(x,y)\\ -\frac{\partial H}{\partial x}(x,y)\end{pmatrix}=\begin{pmatrix}2y\\ -6x^2+6x\end{pmatrix}$$

↳ defines a Hamiltonian v.f.

$$\begin{aligned}\dot{x}&=2y\\ \dot{y}&=-6x^2+6x\end{aligned} \longrightarrow \varphi_t \text{ the flow of } \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=f(x,y)$$

$\varphi_t=(x(t),y(t))$

$$\frac{d}{dt}H(x(t),y(t))=\frac{d}{dt}\left(2x(t)^3-3x(t)^2+y(t)^2\right)$$
$$=6x(t)^2\cdot\dot{x}(t)-6x(t)\cdot\dot{x}(t)+2y(t)\cdot\dot{y}(t)$$
$$=(6x(t)^2-6x(t))2y(t)+2y(t)(-6x^2(t)+6x(t))=0$$

The level curves of H are invariant.

1) $y=0, x=0$ $\quad \dot{x}=0$, $\dot{y}=0$ $\quad Df(x,y)=\begin{pmatrix}0&2\\-12x+6&0\end{pmatrix}$ at 1) $Df(0,0)=\begin{pmatrix}0&2\\6&0\end{pmatrix}$, $\lambda=\pm\sqrt{12}$

2) $y=0, x=1, -6x(x-1)=0$ at 2) $Df(1,0)=\begin{pmatrix}0&2\\-6&0\end{pmatrix}$ $\lambda=\pm\sqrt{12}$

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p hyp. fixed point of f, φ_t restricted to a neighborhood of p the flow of $\dot{x}=f(x)$, is congugate to the flow of the linearized equation $\psi(t,y)=e^{tA}y$ in a neighborhood of the origin $\{A=Df(p)\}$

<u>Example</u> (What if there is decay but no hyperbolicity?)
(asymptotic stability)

$f:\mathbb{R}^2 \to \mathbb{R}^2$

$$f(x,y)=\begin{pmatrix}-y-x^3\\ x-y^3\end{pmatrix} \qquad \begin{aligned}\dot{x}&=-y-x^3\\ \dot{y}&=x-y^3\end{aligned}$$

$(x,y)=(0,0)$ is the only fixed point

$$Df(x,y)\Big|_{(0,0)}=\begin{pmatrix}-3x^2 & -1\\ 1 & -3y^2\end{pmatrix}\Bigg|_{(0,0)}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

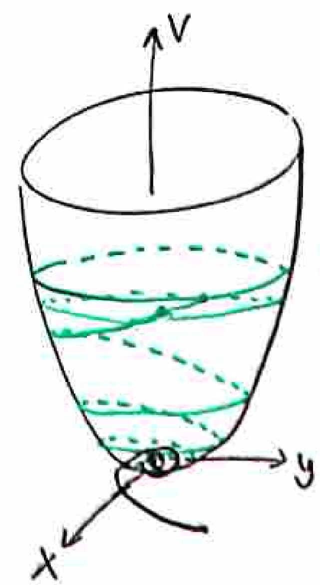
$\lambda^2+1=0$
$\lambda_{1,2}=\pm i$
real part is zero.

The origin is not a hyperbolic fixed point.
However, let's notice that we have <u>asymtotic stability.</u>

We consider a function $V:\mathbb{R}\to\mathbb{R}$

$V(x,y)=x^2+y^2$, $\varphi_t=(x(t),y(t))$ the flow of $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=f(x$

$$\begin{aligned}\frac{d}{dt}V(x(t),y(t)) &= 2x(t)\cdot\dot{x}(t)+2y(t)\cdot\dot{y}(t)\\ &= 2x(t)(-y(t)-x(t)^3)+2y(t)(x(t)-y^3(t))\\ &= -2x^4(t)-2y^4(t)<0\end{aligned}$$



This is an example of a Lyapunov function.

has to go downhill
we descend on the paraboloid.

Then the solution has to go to to the origin $(0,0)$

Every solution does the same $\Rightarrow (0,0)$ is asymtotic stable

It's globally asymp. stable.
This is another central behaviour.