Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

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Extension of solutions

$\frac{d}{dt}x = f(x,t)$, $x(t_0) = x_0$ (IVP)

<u>Thm</u> (Picard-Lindelöff)

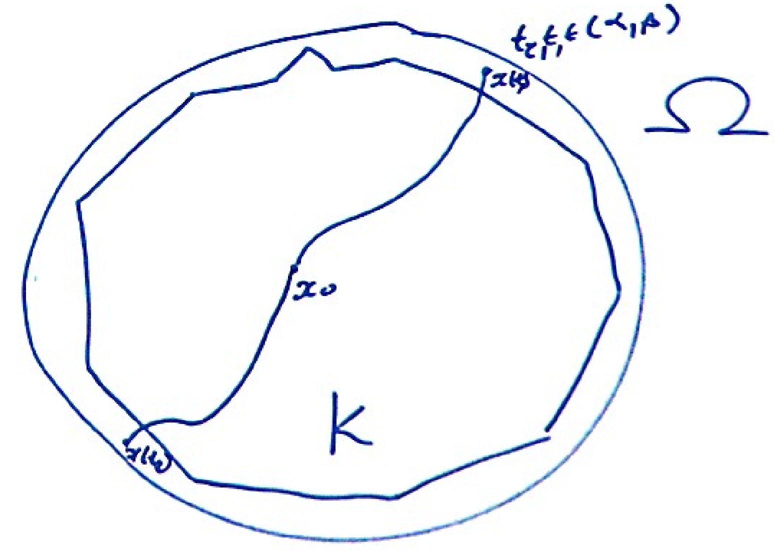
- $f : \Omega \times J \to \mathbb{R}^n$
- f is continuous
- f is Lipschitz w.r.t. x

$\exists!\ x(t)$ s.t. $x(t_0) = x_0$. $t \in J_0$

<u>Thm</u> (Extension of solutions)

Let $\Omega \subset \mathbb{R}^n$, $J \subset \mathbb{R}$ open sets such that the open interval $(\alpha, \beta) \subset J$ and $x_0 \in \Omega$. $f : \Omega \times J \to \mathbb{R}^n$ is C^1 and the maximal interval of existence of the soln of IVP is $\alpha < t < \beta < \infty$, then $\forall K \subseteq \Omega$ compact $\exists\, t \in (\alpha, \beta)$ such that $x(t) \notin K$.

In particular, $|x(t)|$ goes to ∞ or $x(t)$ approaches the boundary of Ω when $t \to \alpha$ or β.



Peano's existence theorem

$\frac{d}{dt}x = f(x,t), \quad x(t_0) = x_0 \quad$ (IVP)

Thm (Picard-Lindelöff)

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Thm (Peano)

$\Omega \subset \mathbb{R}$, $J \subset \mathbb{R}$ open sets. $f: \Omega \times J \to \mathbb{R}$

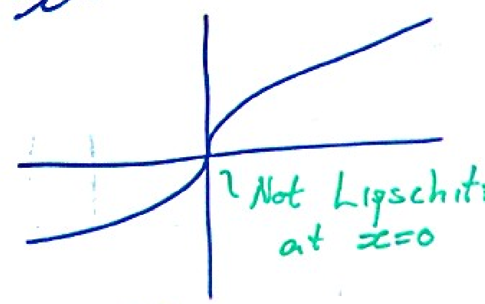
- f is continuous.

Then the (IVP) has a local solution $x: \Omega \times J_0 \to \mathbb{R}$

(Note that x is not necessarily unique!)

Example (couter-example)

$\dot{x} = x^{1/3}$, $x(0) = 0$



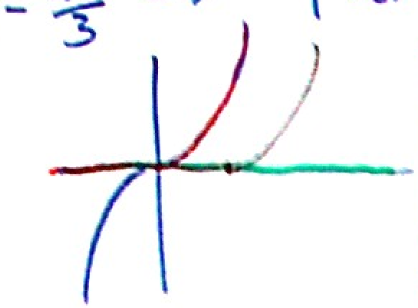
Notice that $x(t) = 0$ is a solution.

If $x(t) \neq 0$

$$\frac{d\phi(t)}{dt} \Big/ \phi(t)^{1/3} = 1 \iff \frac{d}{dt}\left(\frac{3}{2}\phi^{2/3}(t)\right) = 1$$

$$\frac{3}{2}\int_0^t \phi^{2/3}(s)\,ds = t$$

$$\left[\phi^{2/3}(t)\right] = \frac{2t}{3} \Rightarrow \phi(t) = \left(\frac{2t}{3}\right)^{3/2}$$



HW It is possible to show that this problem has uncountably many solutions satisfying $\dot{x} = x^{1/3}$, $x(0) = 0$

Flow property

Flow property

An autonomous differential equation

$$\frac{dx}{dt} = f(x) \leftarrow \text{does not depend explicitly on time}$$

$$\frac{dx}{dt} = \sin(x) \quad \text{autonomous,} \qquad \frac{dx}{dt} = x^2 + t \quad \text{is non-autonomous.}$$

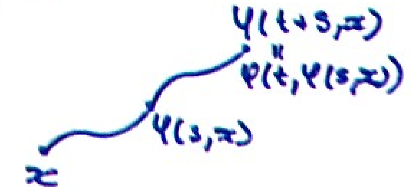
We write the solution $x(t) = \phi(t, x_0)$

- The solution satisfies a group action.

Def

A flow on a set X is a group action of the additive group of real numbers on X, $\varphi: \mathbb{R} \times X \to X$ so that $\forall x \in X$ and $s, t \in \mathbb{R}$,

1) $\varphi(0, x) = x$
2) $\varphi(t, \varphi(s, x)) = \varphi(t+s, x)$



Homogeneous Linear Systems

Now let's think about non-autonomous problems of the form

$$\dot{x} = A(t,\mu)x + \underbrace{g(t,\mu)} = f(x,t)$$

where $A(t,\mu)$ is an $n \times n$ matrix and g is called the non-homogeneous part.

$x_1, x_2 \in \mathbb{R}^n$

$\| f(x_1,t) - f(x_2,t) \|$

$= \| A(t,\mu)x_1 - A(t,\mu)x_2 \| = \| A(t,\mu)[x_1 - x_2] \|$

$\leq \| A(t,\mu) \| \, \| x_1 - x_2 \|$

If $A(t,\mu)$ is nice then f is Lipschitz.
and we have a unique solution

1. There is a more or less complete understanding of the linear part $A(t,\mu)x$.
2. Linear systems can be used to study non-linear systems in general close to special solutions (fixed point, periodic orbit).

Homogeneous Linear Systems

$$\dot{x} = A(t)x \, , \quad A(t) \text{ is an } n \times n \text{ matrix.}$$

•) The solution exists (and is unique) as long as the function $A(t)$ is continuous ($\|A(t)\|$ is bounded in a compact set of times.)

••) Remember that $X = \mathcal{L}(\mathbb{R}^n)$ the set of linear operators in $\mathbb{R}^n$ is a Banach space.
Their elements are represented by $n \times n$ matrices.
and the operator norm is,

$$\|A\| = \sup_{\|v\|_{\mathbb{R}^n} = 1} \|Av\|_{\mathbb{R}^n}$$

Here $\|\cdot\|_{\mathbb{R}^n}$ is the euclidean norm of $\mathbb{R}^n$.

•••) Notice that (or wait for the homework)
$A, B \in X \qquad \|AB\| \leq \|A\| \, \|B\|$ (Unitary Banach algebra)

Linear Systems

<u>Thm</u> (Chicone '06, Thm 2.4)

If $t \to A(t)$ is continuous in the interval $\alpha < t < \beta$ and $t_0 \in (\alpha, \beta)$ then the solution of the IVP,

$$\dot{x}(t) = A(t)x(t)$$
$$x(t_0) = x_0$$

is defined in (α, β).

<u>Pf</u> Already given.

$\Rightarrow$ The solutions of the homogeneous linear system exist as long as A is continuous.

Let's see this fact.

In order to see that the solutions exist inside the interval suppose that $t_0 < b < \beta$ and assume that $|x(t)| \to \infty$ when $t \to b$

$$x(t) = x_0 + \int_{t_0}^{t} A(s)x(s)\,ds$$

By continuity $\exists\ M > 0$ s.t $\|A(s)\| < M$ $\forall s \in [t_0, b]$

$$|x(t)| \leq |x_0| + \int_{t_0}^{t} M\,|x(s)|\,ds$$

$\alpha = |x_0|$
$\varphi = |x(t)|$
$\psi = M$

$\overset{\text{Grönwall's ineq.}}{\Longrightarrow}$ $|x(t)| \leq |x_0| e^{M(t-t_0)} < \infty$

This is a contradiction to the assumption that $|x(t)| \xrightarrow[t \to b]{} \infty$

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Superposition Principle

Superposicion principle.

Prop
If $x_1(t)$ and $x_2(t)$ are solutions of
$$\dot{x} = A(t)x \quad \text{defined for } t \in (a,b)$$
then $\underline{\lambda_1 x_1(t) + \lambda_2 x_2(t)}$ is a solution for $\lambda_1, \lambda_2 \in \mathbb{R}$.

Proof
$$u(t) = \lambda_1 x_1(t) + \lambda_2 x_2(t)$$
$$\left(\frac{d}{dt}\right) \rightarrow \dot{u}(t) = \lambda_1 \dot{x}_1(t) + \lambda_2 \dot{x}_2(t) = \lambda_1 A(t) x_1 + \lambda_2 A(t) x_2(t)$$
$$= A(t)\left(\lambda_1 x_1(t) + \lambda_2 x_2(t)\right) = A(t)\, u(t)$$
So $u(t)$ is also a solution. //

We would like to know if the new solution has new information about the problem.
$\longrightarrow$ The key is the flow preserves the rank of the

Prop
Let $\{x_i(t)\}_{i=1}^{n}$ be a set of solutions of $\dot{x} = A(t)x$ in the interval J and
$$y(t) = \sum_{i=1}^{n} a_i x_i(t) \quad \text{for } a_i \in \mathbb{R}.$$
If $y(t_0) = 0$ for $t_0 \in J$ then
$$y(t) = 0 \quad \text{for all } t \in J.$$

Proof
This follows from the uniqueness. //

Linear independence

Cor

If $\{x_i(t_0)\}_{i=1}^n$ are linearly independent then $\{x_i(t)\}_{i=1}^n$ are also l. i. $\forall t \in J$.

$A(t)X$

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