Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

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$\dot y = Df(0)y + h(y,t)$

Thm

- $\dot x = Ax + g(x,t)$, $x(t_0)=x_0$
- A has eigenvalues with negative real parts.
- $a>0$, $k>0$, $\|g(x,t)\| \le k\|x\|$ $\forall \|x\|<a$.

$\Rightarrow$ $C, b, \alpha>0$ (independent of t_0) constants s.t.

$$\|x(t)\| \le C\|x_0\| e^{-\alpha(t-t_0)}$$

for $t_0 \le t$ if $\|x_0\| < b$.

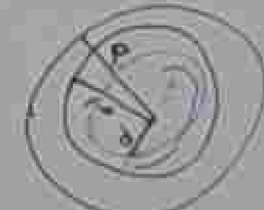
Note. The zero soln is asymptotically stable.

Pf $\exists\, C>1$ and $\lambda>0$ s.t. $\|e^{tA}\| \le Ce^{-\lambda t}$

Let $\delta>0$ s.t. $\delta<a$ and $Ck\delta-\lambda<0$

Define $\alpha := \lambda - Ck\delta > 0$, $b := \delta/C$

$\alpha>0$, $0<b<\delta<a$



If $\|x_0\| < b$ ...

If $\|x_0\|$ ...

so that $\|x(t)\| < \delta$

$t \in J$ since $\|x(t)\|<a$ then

$\|g(x(t),t)\| \le k\delta\|x(t)\|$

Duhamel

$$x(t) = e^{A(t-t_0)}x_0 + e^{A(t-t_0)}\int_{t_0}^{t} e^{A(t_0-s)} g(x(s),s)\,ds$$

$\|g(x(s),s)\| \le k\delta\|x(s)\|$

$$\|x(t)\| \le Ce^{-\lambda(t-t_0)}\|x_0\| + C\int_{t_0}^{t} e^{-\lambda(t-s)} k\delta\|x(s)\|\,ds$$

$$e^{\lambda(t-t_0)}\|x(t)\| \le C\|x_0\| + Ck\delta\int_{t_0}^{t} e^{\lambda(s-t_0)}\|x(s)\|\,ds$$

Gronwall's inequality

$$\Rightarrow e^{\lambda(t-t_0)}\|x(t)\| \le C\|x_0\| e^{Ck\delta(t-t_0)}$$

$\left(\text{times } e^{-\lambda(t-t_0)}\right) \Rightarrow \|x(t)\| \le C\|x_0\| e^{-\alpha(t-t_0)}$

$\boxed{\text{Question: Prove that } J \text{ is in fact } \infty}$

$\dot y = Df(\xi) y + N(y,t)$

Thm

- $\dot x = Ax + g(x,t)$, $x(0)=x_0$
- A has eigenvalues with negative real parts
- $a>0$, $k>0$, $\|g(x,t)\| \le k\|x\|^2$, $\|x\|<a$

$\Rightarrow$ $C, b, \alpha>0$ (independent of t_0) constants s.t.

$$\|x(t)\| \le C\|x_0\| e^{-\alpha(t-t_0)}$$

for $t_0 \ge t$ if $\|x_0\| < b$.

Note. The zero soln is asymptotically stable.

Cor (C')

$f:\mathbb{R}^n \to \mathbb{R}^n$ is smooth, $f(\xi)=0$ and $Df(\xi)$ has eigenvalues with negative real part, then the diff eq $\dot x = f(x)$ has an asymptotically stable fixed point. Moreover, if $-\alpha$ is greater than any of the real parts of the eigenvalues and φ_t is the flow then there is a Nbhd U of ξ and a constant $C>0$ so that,

$$\|\varphi_t(x)-\xi\| \le C\|x\| e^{-\alpha t}, \quad x\in U, \quad t\ge 0$$

Pf (of Cor)

f is smooth so $\exists$ B open ball centered at the origin s.t. $\exists$ $k>0$ and

$$\sup_{\xi\in[0,1]} \|D^2 f(\xi x)\| < k, \quad x \in B$$

By the mean value theorem,

$$\|Df(sx) - Df(0)\| \le \|D^2 f(\eta)\| \, |sx - 0| \quad (\eta \in [0,sx])$$

$$\le |sx| \sup_{\xi\in[0,1]} \|D^2 f(\xi s x)\| \le |x| \sup_{\xi\in[0,1]} \|D^2 f(\xi x)\|$$

$$\le k|x|$$

$\dot x = Df(0)x + g(x)$, $g(x) := \int_0^1 (Df(sx) - Df(0))\,ds$

Now, $\|g(x)\| \le \sup_{s\in[0,1]} |x| \, \|Df(sx)-Df(0)\| \le k\|x\|^2$

By the previous thm //

Simple fixed points

$f: \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field.

Define $p \in \mathbb{R}^n$ is a simple fixed point (s.f.p) if $f(p)=0$
$Df(p)$ is an isomorphism

$$\boxed{\begin{array}{l}\textbf{Prop}\ \text{A simple fixed point is isolated.} \\ \text{The number of s.f.p in a compact set is finite.}\end{array}}$$

Defn f_λ is C^r, $r \geq 1$, is a family of vector fields that pass through f at $\lambda = 0$

$f_\lambda : (-a,a)^\ell \times \mathbb{R}^n \to \mathbb{R}^n$

$(\lambda, x) \mapsto f_\lambda(x)$

$f_0(x) = f(x)$

Prop f is a C^r v.f. and f_λ is a family passing through f.
If p is a s.f.p. of f then $\exists\, \delta > 0$
a neighborhood of p and a C^r function
$(-\delta, \delta)^\ell \to V$ such that $p_0 = p$ and
$\lambda \mapsto p_\lambda$
for every $\lambda \in (-\delta, \delta)$ the point p_λ
is the only fixed point of the v.f. $f_\lambda : \mathbb{R}^n \to \mathbb{R}^n$
and it is simple.

$\boxed{\text{Pf. Volunteers?}}$

Prop Let $\Phi = \varphi_1$ the time 1 flow of f.
The derivative $D\Phi(p)$ is given by $D\Phi(p) = e^{Df(p)}$

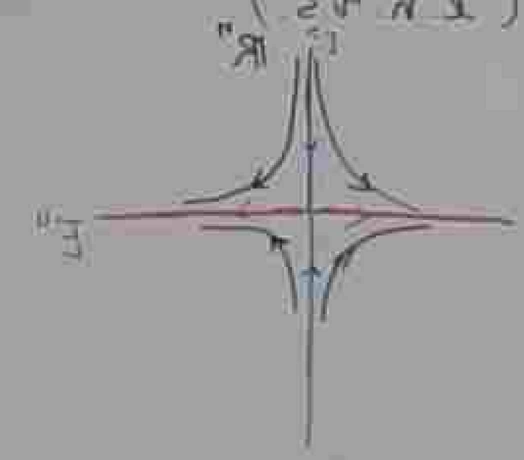
Pf (corolary of the proof by Victor.

Hyperbolic fixed points

Def

We say that $p \in \mathbb{R}^n$ is a hyperbolic fixed point of f if $Df(p) : \mathbb{R}^n \to \mathbb{R}^n$ has no eigenvalues with zero real part.

In that case we say that $x' = Ax$ with $A := Df(p)$ is a linear hyperbolic vector space (l.h.v.s.)



If A is a l.h.v.s. then there exists a decomposition of $\mathbb{R}^n = E^s \oplus E^u$

- E^s, E^u are invariant.
- the eigenvalues of $A|_{E^s}$ have negative real parts.
- the eigenvalues of $A|_{E^u}$ have positive real parts.

We will call E^s the stable linear space and E^u the unstable l.s.

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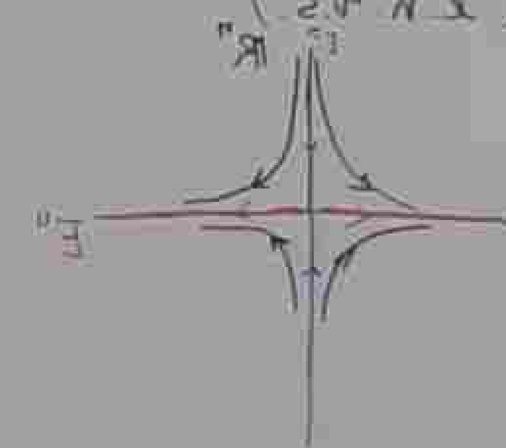
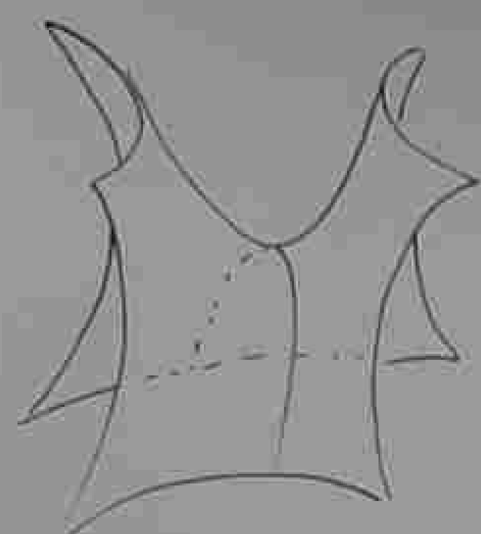


Classification of hyperbolic fixed points.

- Attractor: A has only e.v. with neg real part
- Repeller: A has e.v. with positive real parts
- Saddle: A has e.v. with pos. & neg r.p.

Saddle — "Silla de montar"



Hyperbolic fixed points

Corollary

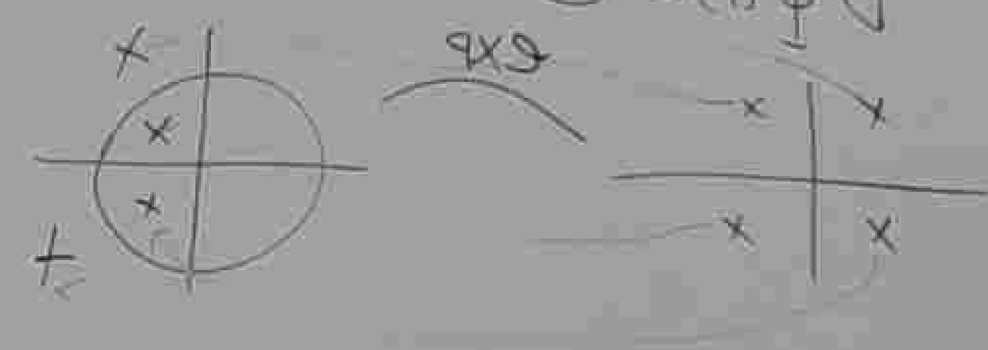
Let $\Phi = \varphi_1$ be the time 1 flow of f.

Where $p \in \mathbb{R}$ is a hyperbolic fixed point.

$\iff$ $D\Phi(p)$ does not have eigenvalues on the unit circle

Pf

$D\Phi(p) = e^{Df(p)}$



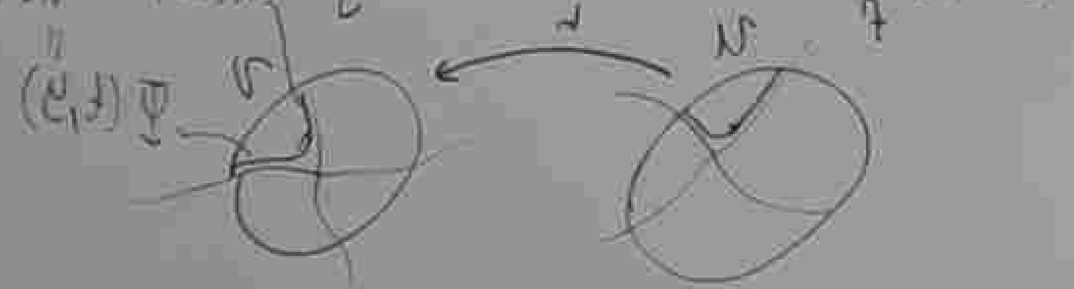
Def

The flows $\varphi(t,x)$ & $\psi(t,y)$ of 2 v.f.s $f: U \subset \mathbb{R}^n \to \mathbb{R}^n$ & $g: V \subseteq \mathbb{R}^n \to \mathbb{R}^n$ are conjugate if there exists a homeomorphism $h: U \to V$ such that

$h(\varphi(t,x)) = \Psi(t, h(x))$

$\forall x \in U$ & every t so that φ_t is defined

In the case of h.f.p.s.



Grobman - Hartman theorem
1959 1960

(Note that Poincaré already knew the result for analytic vector fields [nineteenth century])

If p is a hyp. fixed point of the v.f. f ($f(p)=0$) then the flow φ_t of $\dot{x} = f(x)$ restricted to a neighborhood of p is conjugate to the flow ψ_t of linear diff. eq. restricted to a nbhd of the origin $\quad \psi(t,y) = e^{tA}y \quad (A := Df(p))$

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The flows $\varphi(t,x)$ & $\psi(t,y)$ of 2 v.f.s $f: U \subset \mathbb{R}^n \to \mathbb{R}^n$ & $g: V \subseteq \mathbb{R}^n \to \mathbb{R}^n$ are conjugate if there exists a homeomorphism $h: U \to V$ such that

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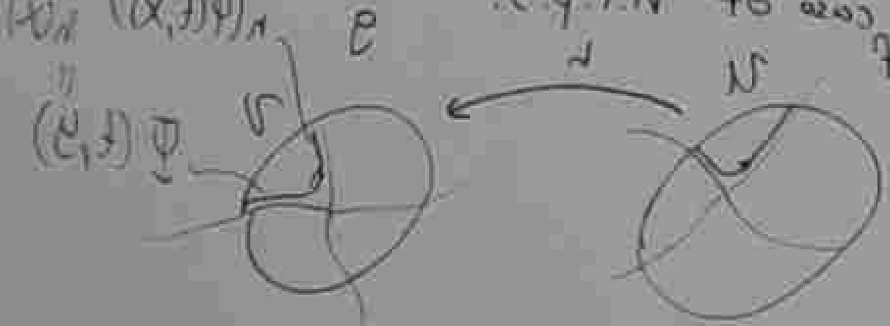

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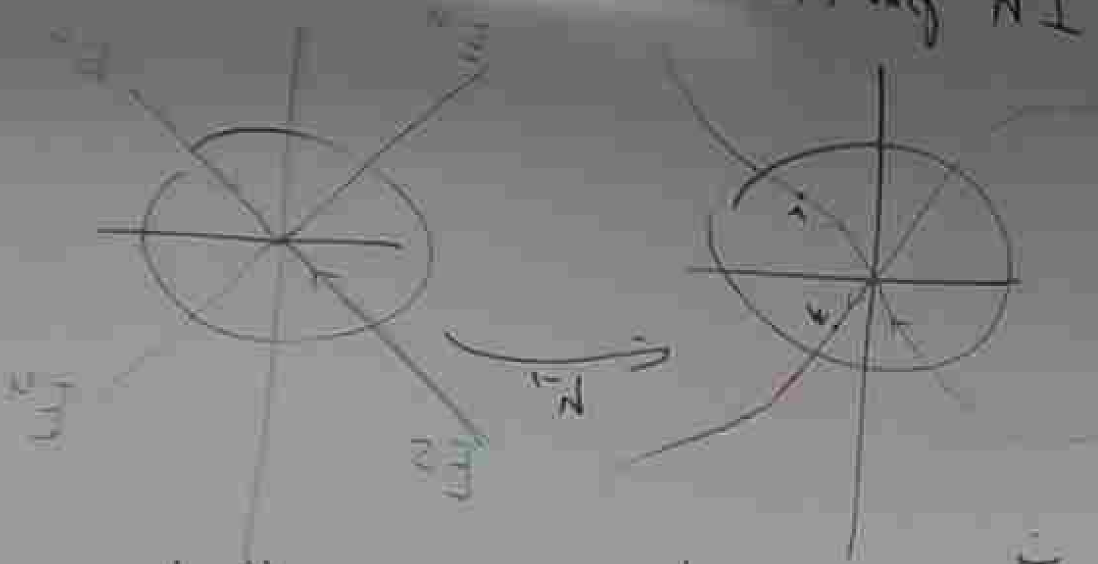
$h(\varphi_t(x)) = e^{tA}y$, $y = h(x)$

In particular we have that



There exist curves so that the dynamics satisfy limit properties.

$\lim_{t\to\infty} \varphi_t(x) = p$ } define stable and unstable

$\lim_{t\to-\infty} \varphi_t(w) = p$ } local manifolds.