Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

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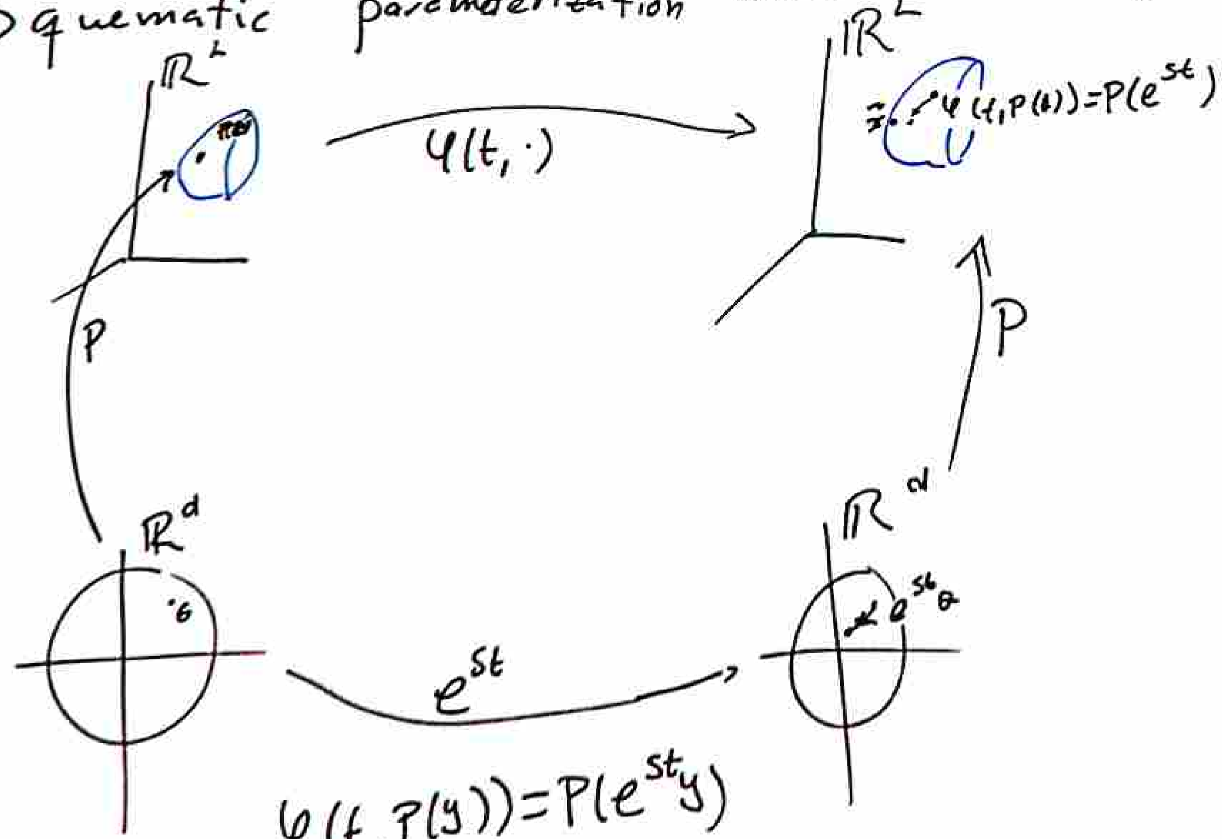
Squematic parameterization method (stable manifold)



$$\varphi(t,P(y)) = P(e^{St}y)$$

$$\dot{x} = f(x)$$
$$x(0)=x_0$$

Lemma (Parameterization lemma)

The function $P : B \subset \mathbb{R}^{L-d} \longrightarrow \mathbb{R}^L$ satisfies the conjugacy equation $\varphi(t,P(y)) = P(e^{St}y)$

$\Longleftrightarrow$ P is a solution to the PDE

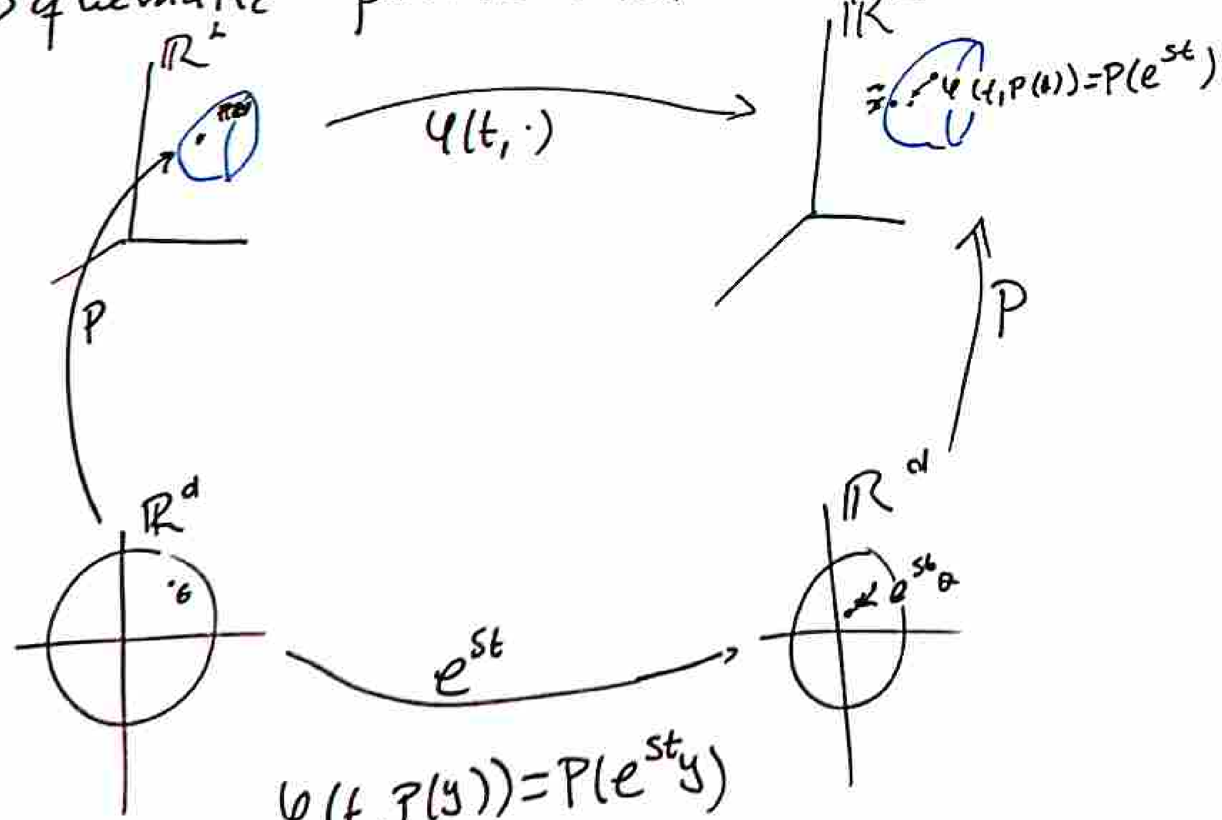
$$DP(y)Sy = f(P(y)), \quad \forall y \in B.$$

Proof ($\Longleftarrow$) P satisfies $DP(y)Sy = f(P(y))$ in B. Fix $y \in B$ and $t>0$. We define $\gamma_y : [0,t] \longrightarrow \mathbb{R}^L$ given by $\gamma_y(t) = P(e^{St}y)$. Then $\gamma_y(0) = P(y)$ and

$$\frac{d}{dt}\gamma_y(t) = DP(e^{St}y)Se^{St}y = f(P(e^{St}y))$$
$$= f(P(\gamma_y(t))) \quad \Longrightarrow \quad \dot{\gamma}_y(t) = f(P(\gamma_y(t)))$$
$$\gamma_y(0) = P(y)$$

$$\Longrightarrow \gamma_y(t) = \varphi(t,\gamma_y(0)) = \varphi(t,P(y))$$
$$\varphi(t,P(y)) = P(e^{St}y)$$

Squematic parameterization method (stable manifold)



$$\varphi(t,P(y))=P(e^{st}y)$$

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Lemma (Parameterization lemma)

The function $P: B \subset \mathbb{R}^{L-d} \longrightarrow \mathbb{R}^L$ satifies the conjugacy equation $\varphi(t,P(y))=P(e^{st}y)$

$\iff$ P is a solution to the PDE

$$DP(y)Sy = f(P(y)), \quad \forall y \in B.$$

Proof ($\Rightarrow$) Suppose that P is s.t. $\varphi(t,P(y))=P(e^{st}y)$ $\forall y\in B$. Let's fix a $y\in B$ and differenciate both sides w.r.t. t

$$\frac{d}{dt}\varphi(t,P(y)) = f(P(y)) = \frac{d}{dt}P(e^{st}y) = DP(e^{st}y)Se^{st}y$$

Taking the limit as $t\to 0$.

$$f(P(y)) = DP(y)Sy \quad \forall y \in B \quad //$$

To explain 3) we assume that S has real eigenvalues and it is diagonalizable.

- $L-d = M$ and denote the eigenvalues of S by $\lambda_1, \dots, \lambda_M$ and for $m = 1, \dots, M$ let ξ_m be the unitary eigenvector associated to λ_m

$\{\xi_m \mid m = 1, \dots, M\}$ is a basis of $\mathbb{R}^M$ and in this bases the semigroup takes the form e^{st} with

$$S = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_M \end{pmatrix}$$

Let $E = [r_1 \xi_1 | r_2 \xi_2 | \dots | r_M \xi_M]$ where $r_m > 0$ with columns given by the eigenvectors.

↳ an $M \times M$ matrix

→ We impose the restriction that $DP(0) = E$

"The stable space is tangent to the manifold at the fixed point".

Lemma (Parameterization lemma)

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$$DP(y)\, S\, y = f(P(y)), \quad \forall y \in B.$$

Notes 1) By the Lemma if we solve the PDE for P then we obtain a parameterization of the stable manifold.

2) It turns out that the PDE can be solved by a Banach contraction argument and obtain an infinite amount of solutions.

3) We will add a restriction to have a unique solution. Namely that the manifold is tangent to stable space at the fixed point.

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Lemma (Parameterization lemma)

The function $P : B \subset \mathbb{R}^{L-d} \to \mathbb{R}^L$ satisfies the conjugacy equation $\varphi(t, P(y)) = P(e^{st} y)$

$\iff$ P is a solution to the PDE

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Note that can be writen explicitly

$$\boxed{\lambda_1 y_1 \frac{\partial}{\partial y_1} P(y_1, \dots, y_M) + \dots + \lambda_M y_M \frac{\partial}{\partial y_M} P(y_1, \dots, y_M) = f(P(y_1, \dots, y_M))}$$

$y = (y_1, \dots, y_M) \in B$

Example (Lorenz equation)

$$\dot{x} = f(x) = \begin{pmatrix} \sigma(x_2 - x_1) \\ \rho x_1 - x_2 - x_1 x_3 \\ -\beta x_3 + x_1 x_2 \end{pmatrix}$$

It has 3 fixed points

$$0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{x}^{\pm} = \begin{pmatrix} \pm\sqrt{\beta(\rho-1)} \\ \pm\sqrt{\beta(\rho-1)} \\ \rho-1 \end{pmatrix}$$

where $\rho > 1$. The classical values of $\sigma = 10$, $\beta = 8/3$, $\rho = 28$

The eigenvalues at the 0 are real (1 pos. 2 neg)
The eigenvalues at $\tilde{x}^{\pm}$ are 1 negative and the others are a complex conjugate pair (unstable)
Let us compute the 1 dimensional manifolds.

Let ξ be the eigenvector associated to λ.
First, we assume that $\lambda > 0$. Here $M=1$ and P is a 1-dimensional function of a single variable.
We express P in Taylor series

$$P(y) = \sum_{n=0}^{\infty} a_n y^n = \sum_{n=0}^{\infty} \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ a_{3,n} \end{pmatrix} y^n$$

analytic.

where $a = \{a_n \in \mathbb{R}^3\}_{n=0}^{\infty}$

In this case $DP(0) = E$ translates into

$$\begin{pmatrix} a_{1,0} \\ a_{2,0} \\ a_{3,0} \end{pmatrix} = r \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

In general we have that $\frac{\partial P}{\partial y}(y)\lambda y = f(P(y))$

$$\Rightarrow \lambda \sum_{n=0}^{\infty} \begin{pmatrix} (n+1)\, a_{1,n+1} \\ (n+1)\, a_{2,n+1} \\ (n+1)\, a_{3,n+1} \end{pmatrix} y^{n+1} = \sum_{n=0}^{\infty} \begin{pmatrix} \sigma(a_{2,n} - a_{1,n}) \\ \rho a_{1,n} - a_{2,n} - (a_1 * a_3)_n \\ -\beta a_{3,n} + (a_1 * a_2)_n \end{pmatrix} y^n$$

where $(x * y) := \sum_{k=0}^{n} x_{n-k} \cdot y_k$ (this reminds us of Cauchy products

Example (Lorenz equation)

$$\dot{x} = f(x) = \begin{pmatrix} \sigma(x_2 - x_1) \\ \rho x_1 - x_2 - x_1 x_3 \\ -\beta x_3 + x_1 x_2 \end{pmatrix}$$

Matching powers we obtain $F(a) = (F_1(a), F_2(a), F_3(a))$

$$(F_1(a))_n = \begin{cases} a_{1,0} & \text{if } n=0 \\ a_{1,1} - r\xi_1 & n=1 \\ \lambda n a_{1,n} - \sigma(a_{2,n} - a_{1,n}) & n \geq 2 \end{cases}$$

$$(F_2(a))_n = \begin{cases} a_{2,0} & n=0 \\ a_{2,1} - r\xi_2 & n=1 \\ \lambda n a_{2,n} - \rho a_{1,n} + a_{2,n} + (a_1 * a_3)_n & n \geq 2 \end{cases}$$

$$(F_3(a))_n = \begin{cases} a_{3,0} & n=0 \\ a_{3,1} - r\xi_3 & n=1 \\ \lambda n a_{3,n} + \beta a_{3,n} - (a_1 * a_2)_n & n \geq 2 \end{cases}$$

You can obtain explicit expressions for the $a_n = \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ a_{3,n} \end{pmatrix}$ by solving $(F(a))_n$ order by order.

→ H/W (Volunteers) implement the solution of a_n and graph the manifold close to the fixed point.

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Center manifold theorem

What happens if the fixed point is not hyperbolic?
- No Grobman - Hartman theorem.

Assume that there is a fixed point with Stable directions, unstable and non-hyperbolic (real part is zero)

$$\dot{x} = Sx + f_1(x,y,z)$$
$$\dot{y} = Uy + f_2(x,y,z)$$
$$\dot{z} = Cz + f_3(x,y,z)$$

1) Group x, z

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} f_1(x,y,z) \\ f_3(x,y,z) \end{pmatrix}$$

You can adapt the proof of the stable manifold theorem and obtain an α (function) parameterizes a center-stable manifold. Manifold is tangent to the center-stable space. E^{cs}

2) Group y, z

$$\begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} f_2(x,y,z) \\ f_3(x,y,z) \end{pmatrix}$$

Obtain ρ (function) parametrizing the unstable-center manifold, tangent to E^{uc}

3) Notice that the intersection of those two manifolds is also a manifold and it is a graph to the U and S spaces. It is tangent to E^c
It is invariant under the flow.

!) However, this manifold is not unique and does not have the same regularity properties as the stable and ustable manifolds.
Before we had manifolds that are as regular as the v.f. but center manifold could be a lot less regular.

Example (Center manifold)

$$\dot{x} = -x^3$$
$$\dot{y} = -y + x^2$$
} this is analytic

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}-x^3\\ x^2\end{pmatrix}$$

$E^c = \{x \text{ axis}\}$ $\quad E^s = \{y \text{ axis}\}$

Let $y = h(x)$ be the graph of a center manifold

$$-h(x) + x^2 = \dot{y} = \frac{dh}{dx}(x)\dot{x} = -\frac{dh(x)}{dx}x^3$$

Suppose that h is analytic $h(x) = \sum_{k=2}^{\infty} a_k x^k$, $\frac{dh}{dx}(x) = \sum_{k=2}^{\infty} a_k k x^{k-1}$

$$-\sum_{k=2}^{\infty} a_k x^k + x^2 = -\sum_{k=2}^{\infty} a_k k x^{k+2} = -\sum_{k=4}^{\infty} a_{k-2}(k-2)x^k$$

$a_0 = 0, a_1 = 0$

Order x^2

$-a_2 + 1 = 0$, $a_2 = +1$

Order x^3

$-a_3 = 0$, $a_3 = 0$

Order x^4

$+a_4 = +a_2 \cdot 2 \Rightarrow a_4 = 2$

Odd powers, order x^{2j+1}

$a_{2j+1} = (2j-1)\, a_{2j-1} = 0 \quad \forall j$

Even powers, order x^{2j}

$a_{2j} = (2j-2)\, a_{2j-2}$

$a_6 = 4 \cdot 2 = 8$

$a_8 = 6 \cdot 4 \cdot 2 = 48$

$a_{10} = 8 \cdot 6 \cdot 4 \cdot 2 = 2^4 (4!)$ The coefficients are growing very fast.

The series for h does not converge for any number different from zero.

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$-a_2 + 1 = 0$, $a_2 = +1$

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Order x^4

$+a_4 = +a_2 \cdot 2 \Rightarrow a_4 = 2$

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Example (Center manifold)

$$\left.\begin{aligned}\dot{x} &= -x^3\\ \dot{y} &= -y + x^2\end{aligned}\right\}$$ this is analytic

but h satisfies a diff eq. given by

$$\frac{dh}{dx} = \frac{h}{x^3} - \frac{1}{x}$$

$$e^{1/2x^2}\frac{dh}{dx} - e^{1/2x^2}\frac{h}{x^3} = -\frac{1}{x}e^{1/2x^2}$$

$$\frac{d e^{1/2x^2}}{dx} = -\frac{1}{x^3}e^{1/2x^2}$$

$$\int \frac{d}{dx}\left(e^{1/2x^2}h(x)\right)dx = -\int \frac{1}{x}e^{1/2x^2}dx$$

$$e^{1/2x^2}h(x) = -\int \frac{e^{1/2x^2}}{x}dx$$

$$h(x) = -e^{-1/2x^2}\int \frac{e^{1/2x^2}}{x}dx = e^{-1/2x^2}E_i\left(\frac{1}{2x}\right)$$

E_i is the exponential integral function.

$$E_i(x) = -\int_{-x}^{\infty}\frac{e^{-t}}{t}dt$$

$$= \underset{\text{Euler-Mascheroni}}{\gamma} + \ln x + \underbrace{x + \frac{1}{4}x^2 + \frac{1}{18}x^3 + \frac{1}{96}x^4 + \frac{1}{600}x^5 + \dots}_{\text{analytic}}$$

h is clearly not analytic. !