Mapeo estándar

Sabemos que el flujo de las ecs. de Hamilton es un simplectomorfismo $\phi_{t,t_0}(z)$, $z \in \mathbb{R}^{2n}$, $\in T^n \times \mathbb{R}^n$

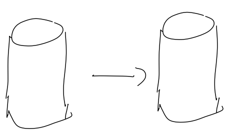
En general, no es posible escribir una fórmula explícita para ϕ.

$$T_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{bmatrix} q \\ p \end{bmatrix} \mapsto \begin{bmatrix} q + p + \varepsilon G(q) \pmod 1 \\ p + \varepsilon G(q) \end{bmatrix}$$

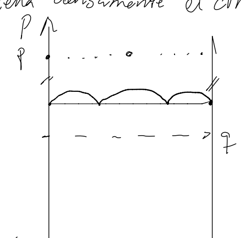
$$G(q) = \frac{1}{2\pi} \sin(2\pi q)$$ función periódica de periodo 1

$$T : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$$



si $p \in \mathbb{Q}$ $p = l/m$, $l \in \mathbb{Z}$, $m \in \mathbb{N}$ la órbita es periódica.

si $p \in \mathbb{R}/\mathbb{Q}$ la órbita llena densamente el círculo.



Caso $\varepsilon = 0$

$$T_\varepsilon \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q + p \pmod 1 \\ p \end{pmatrix}$$

Todos los círculos $p = cte$ son invariantes bajo la dinámica.

$\varepsilon \neq 0$ dinámica más complicada.

$$\begin{pmatrix} q + p + \frac{\varepsilon}{2\pi} \sin(2\pi q) \\ p + \frac{\varepsilon}{2\pi} \sin(2\pi q) \end{pmatrix} = \begin{pmatrix} q + p + \varepsilon G(q) \\ p + \varepsilon G(q) \end{pmatrix}$$

Veamos que el mapeo es simpléctico.

$$DT\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 1 + \varepsilon G'(q) & 1 \\ \varepsilon G'(q) & 1 \end{pmatrix}$$

$$DT\begin{pmatrix} q \\ p \end{pmatrix}^T J \, DT\begin{pmatrix} q \\ p \end{pmatrix} =$$

$$= \begin{pmatrix} 1 + \varepsilon G'(q) & \varepsilon G'(q) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 + \varepsilon G'(q) & 1 \\ \varepsilon G'(q) & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \varepsilon G'(q) & \varepsilon G'(q) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon G'(q) & 1 \\ -(1 + \varepsilon G'(q)) & -1 \end{pmatrix}$$

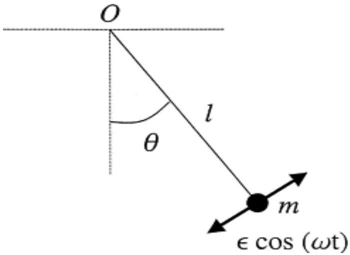
$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J$$

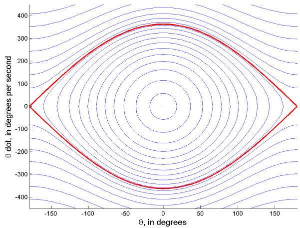
El mapeo estándar es simpléctico.

$$\det\left(Df\begin{pmatrix} q \\ p \end{pmatrix}\right) = 1 + \varepsilon G'(q) - \varepsilon G'(q) = 1$$

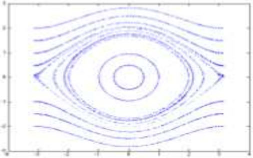
Péndulo perturbado

$$\dot{x} = y$$
$$\dot{y} = -\omega^2 \sin(x) + \epsilon \cos(\omega t)$$

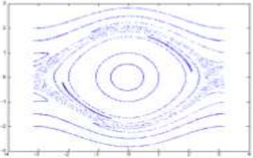




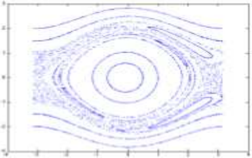
El espacio fase se “destruye”.



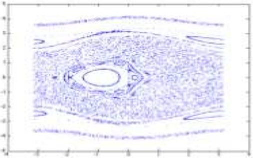
(a) $\epsilon = 0$



(b) $\epsilon = 0{,}01$

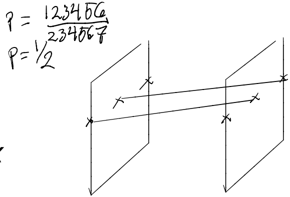


(c) $\epsilon = 0{,}1$



(d) $\epsilon = 1{,}0$

Objetos invariantes en mapeos



- Puntos fijos

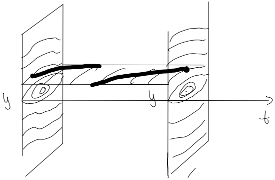
$$f(x) = x$$

- Órbitas periódicas

$$f^n(x) = x$$

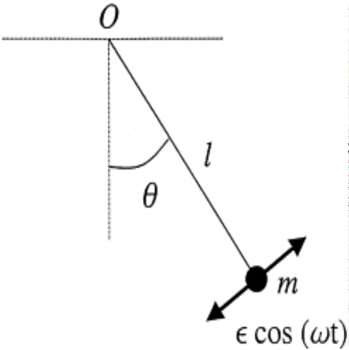
- Círculos invariantes

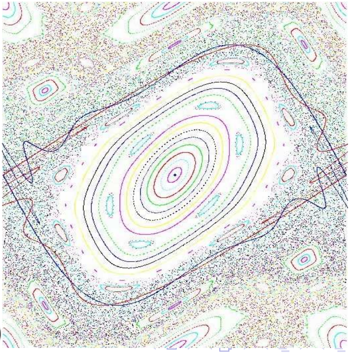
$$f(\text{círculo}) = \text{círculo}$$



Péndulo perturbado

$$H(y,x) = \frac{1}{2}y^2 + \varepsilon V(x) \times \sum_{n=-\infty}^{\infty} \delta\left(\frac{t}{T} - n\right)$$



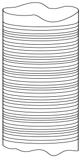


Puntos fijos, órbitas periódicas y círculos invariantes

Mapeo integrable.

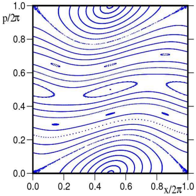
$$
\begin{aligned}
y' &= y \\
x' &= x + \omega(y) \pmod 1
\end{aligned}
$$

ω es el número de rotación.



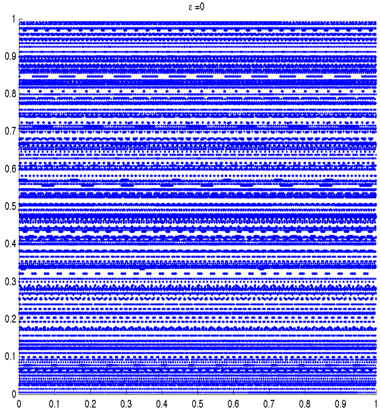
Mapeo estándar.

$$
\begin{aligned}
y' &= y + \frac{\varepsilon}{2\pi}\sin(2\pi x) \\
x' &= x + y' \pmod 1
\end{aligned}
$$

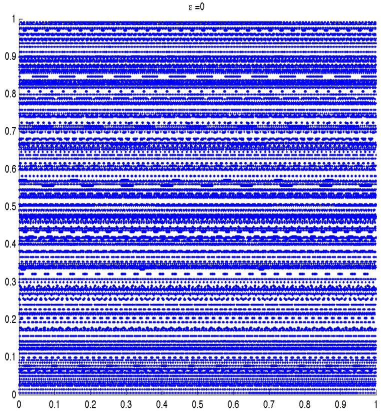


Moviendo ε

$$y' = y + \frac{\varepsilon}{2\pi}\sin(2\pi x)$$
$$x' = x + y' \pmod 1$$



$$y' = y + \frac{\varepsilon}{2\pi}\sin(2\pi x)$$
$$x' = x + y' \pmod 1$$



$$y' = y + \frac{\varepsilon}{2\pi}\sin(2\pi x)$$
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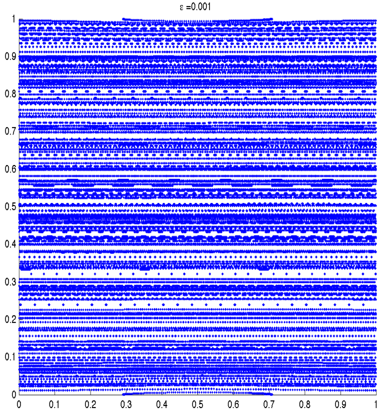
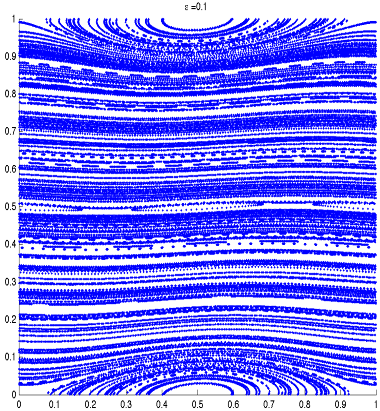


$$y' = y + \frac{\varepsilon}{2\pi}\sin(2\pi x)$$
$$x' = x + y' \pmod 1$$



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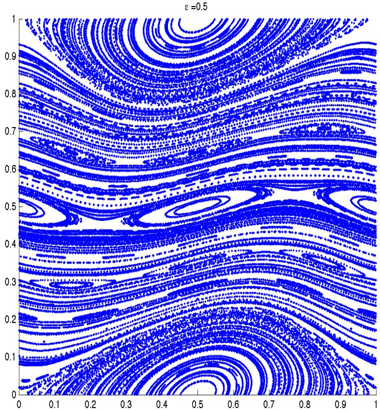

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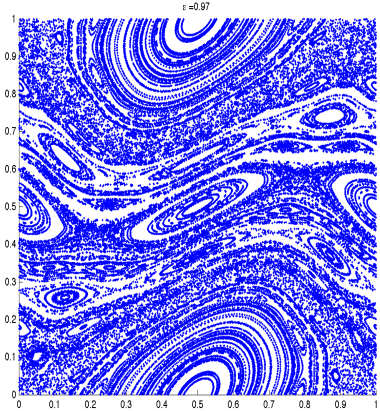
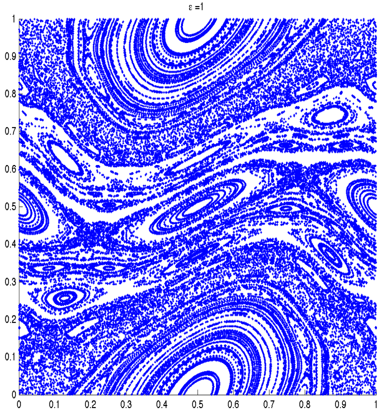


$$y' = y + \frac{\varepsilon}{2\pi} \sin(2\pi x)$$
$$x' = x + y' \pmod 1$$



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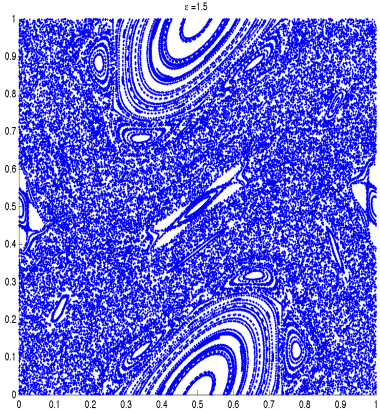

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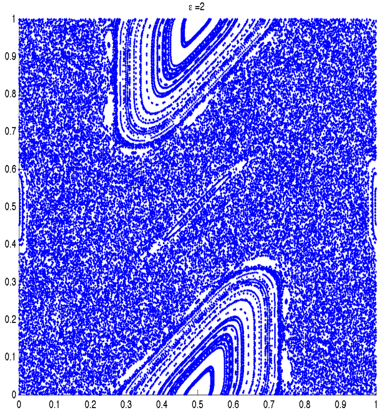

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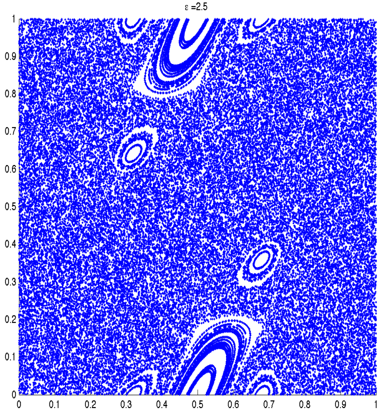

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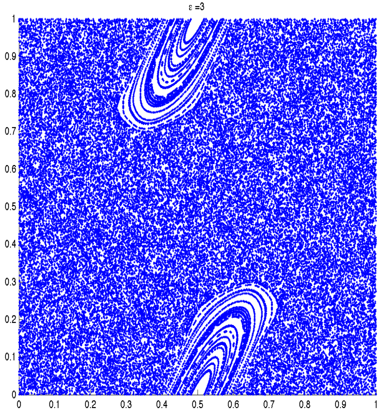

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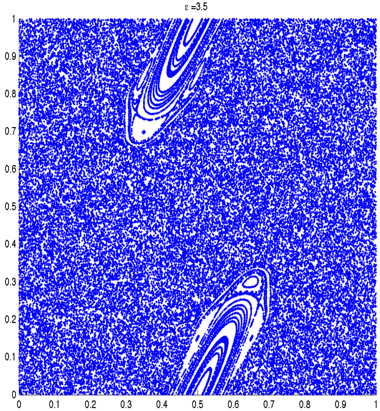

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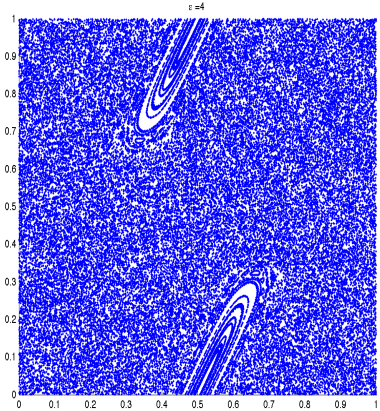

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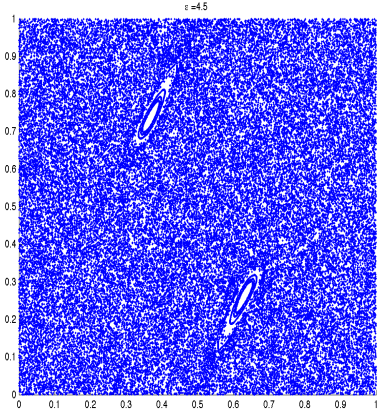

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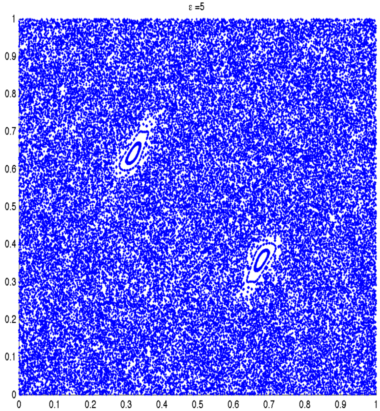

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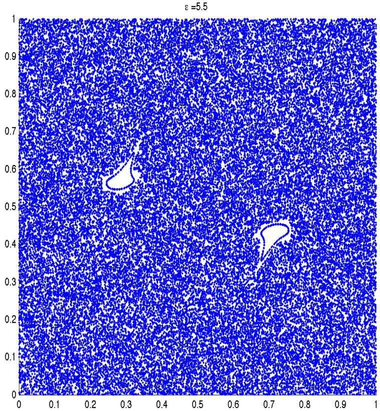

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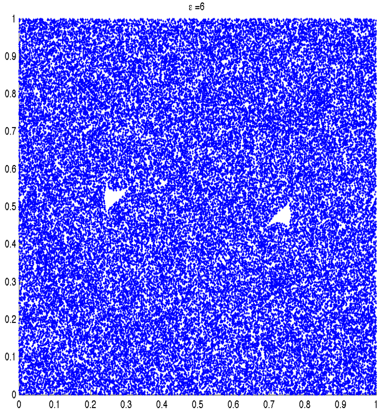
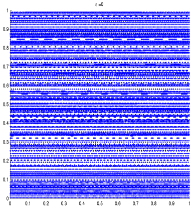
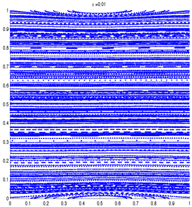


Teorema KAM

Las órbitas quasi-periódicas persisten cerca del caso integrable





Series de Lindstedt

$$T_\varepsilon \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q + p - \varepsilon V'(q) \pmod 1 \\ p - \varepsilon V'(q) \end{pmatrix}$$

con $V(q) = V(q+1)$ es un potencial suave.

$$q_{n+1} = (q_n + p_{n+1}) \pmod 1$$
$$p_{n+1} = p_n - \varepsilon V'(q_n)$$

$$p_{n+1} = q_{n+1} - q_n$$
$$p_n = q_n - q_{n-1}$$

$$q_{n+1} - q_n = q_n - q_{n-1} - \varepsilon V'(q_n)$$

$$q_{n+1} - 2q_n + q_{n-1} = -\varepsilon V'(q_n)$$

$$\mathcal{L}(q) = \sum_{n \in \mathbb{Z}} \frac{1}{2}(q_{n+1} - q_n)^2 + \varepsilon V(q_n)$$

Equilibrios $\dfrac{\partial \mathcal{L}}{\partial q_n} = 0$

Frenkel-Kontorova

Deposición de partículas en un sustrato.

Suponemos también que

$$\int_0^1 V'(q)\, dq = 0$$

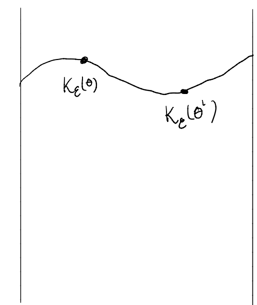
Buscamos soluciones cuasi-periódicas escribiendo una ecuación de invarianza.

→ Buscamos una $K_\varepsilon : \mathbb{T}^1 \to \mathbb{T} \times \mathbb{R}$ tal que

$$T_\varepsilon \circ K_\varepsilon(\theta) = K_\varepsilon(\theta') \quad \text{(invarianza)}$$

Condición de cuasi-periodicidad.

$$K_\varepsilon(\theta') = K_\varepsilon(\theta + \omega)$$



$$T_\varepsilon \circ K_\varepsilon(\theta) = K_\varepsilon(\theta + \omega) \quad \text{(Inv)}$$

$$K(\theta; \varepsilon)$$

Lindstedt

$$K_\varepsilon(\theta) = \sum_{n=0}^{\infty} \varepsilon^n K_n(\theta)$$

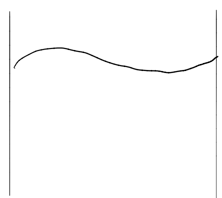
Taylor

$$T_\varepsilon(K_\varepsilon(\theta)) = T_0 \circ K_0 + \varepsilon\left[T_1 \circ K_0 + (DT_0 \circ K_0) K_2 + (DT_1 \circ K_0) K_1\right] + \frac{1}{2}(D^2 T_0 \circ K_0)[K_1, K_1] + \dots$$

En la formulación lagrangiana.

$$K_\varepsilon(\theta) = \begin{pmatrix} \theta + l_\varepsilon(\theta) \\ u_\varepsilon(\theta) \end{pmatrix} \qquad K_\varepsilon : \mathbb{T}^1 \to \mathbb{T}^1 \times \mathbb{R}$$

$$g_\varepsilon : \mathbb{T}^1 \to \mathbb{T}^1$$



$Id(\theta) = \theta$

$v(\theta) = 2\theta \pmod 1$

$= 3\theta \pmod 1$

$= 4\theta$

$= k\theta$

$k \in \mathbb{Z}$

$$g_\varepsilon(\theta+1) = g_\varepsilon(\theta) + 1$$

Escribimos $g_\varepsilon(\theta) = \theta + l_\varepsilon(\theta)$

periódica $l_\varepsilon(\theta+1) = l_\varepsilon(\theta)$

En la formulación lagrangiana,

$$l_\varepsilon : \mathbb{T}^1 \to \mathbb{T}^1$$

$$q = \theta + l_\varepsilon(\theta)$$

(Inv) es equivalente a.

(inv-lag) $$l_\varepsilon(\theta+\omega) + l_\varepsilon(\theta-\omega) - 2 l_\varepsilon(\theta) = -\varepsilon V'(\theta + l_\varepsilon(\theta))$$

¿Dónde está la órbita de un punto q_n?

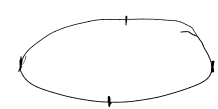
$$q_n = n\omega + l_\varepsilon(n\omega)$$

En la ecuación de invarianza tenemos que si $K_\varepsilon(\theta)$ es una solución,

$$T_\varepsilon(K_\varepsilon(\theta)) = K_\varepsilon(\theta+\omega),$$

entonces la función $\underline{K_\varepsilon(\theta+\tau)}$ con $\tau \in \mathbb{T}^1$

$$T_\varepsilon(K_\varepsilon(\theta+\tau)) = K_\varepsilon(\theta+\tau+\omega).$$



Esto nos dice que hay un # infinito de soluciones. Esto es mejor que no tener solución.

En términos de la función periódica l_ε, tenemos,

$$K_\varepsilon(\theta+\sigma) = \begin{pmatrix} \theta+\sigma + l_\varepsilon(\theta+\sigma) \\ u(\theta+\sigma) \end{pmatrix}$$

Si $l_\varepsilon(\theta)$ es solución, entonces

$\sigma + l_\varepsilon(\theta+\sigma)$ también es una solución.

σ ~~está~~ relacionada con el promedio.

Imponemos que $\int_0^1 l_\varepsilon(\theta)\, d\theta = 0$