Mapeo estándar

Sabemos que el flujo de las ecs. de Hamilton es un simplectomorfismo $\phi_{t,t_0}(z)$, $z \in \mathbb{R}^{2n}$ $\in \mathbb{T}^n \times \mathbb{R}^n$

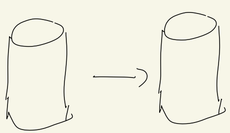
En general, no es posible escribir una fórmula explícita para ϕ.

$$f: \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{bmatrix} q \\ p \end{bmatrix} \mapsto \begin{bmatrix} q + p + \varepsilon G(q) \\ p + \varepsilon G(q) \end{bmatrix}$$

$$G(q) = \frac{1}{2\pi} \sin(2\pi q)$$ función periódica de periodo 1

$$f: S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$$



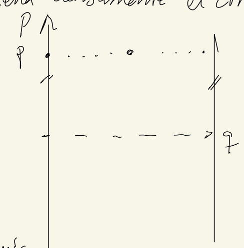
Caso $\varepsilon = 0$

$$f\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q + p \\ p \end{pmatrix}$$

Todos los círculos $p = cte$ son invariantes bajo la dinámica.

si $p \in \mathbb{Q}$ $p = l/m$, $l \in \mathbb{Z}$, $m \in \mathbb{N}$ la órbita es periódica.

Si $p \in \mathbb{R}/\mathbb{Q}$ la órbita llena densamente el círculo.



$\varepsilon \neq 0$ dinámica más complicada.

$$\begin{pmatrix} q + p + \frac{\varepsilon}{2\pi}\sin(2\pi q) \\ p + \frac{\varepsilon}{2\pi}\sin(2\pi q) \end{pmatrix} = \begin{pmatrix} q + p + \varepsilon G(q) \\ p + \varepsilon G(q) \end{pmatrix}$$

Veamos que el mapeo es simpléctico.

$$Df\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 1 + \varepsilon G'(q) & 1 \\ \varepsilon G'(q) & 1 \end{pmatrix}$$

$$Df\begin{pmatrix} q \\ p \end{pmatrix}^T J \, Df\begin{pmatrix} q \\ p \end{pmatrix} =$$

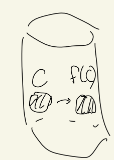
$$= \begin{pmatrix} 1 + \varepsilon G'(q) & \varepsilon G'(q) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 + \varepsilon G'(q) & 1 \\ \varepsilon G'(q) & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \varepsilon G'(q) & \varepsilon G'(q) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon G'(q) & 1 \\ -(1 + \varepsilon G'(q)) & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J$$

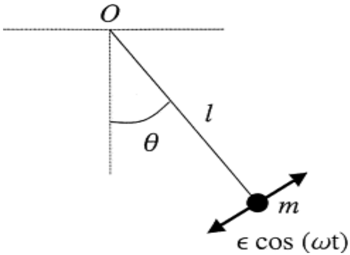
El mapeo estándar es simpléctico.

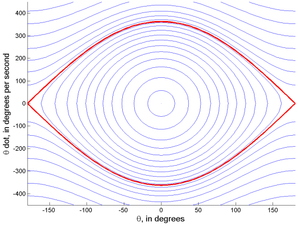
$$\det\left(Df\begin{pmatrix} q \\ p \end{pmatrix}\right) = 1 + \varepsilon G'(q) - \varepsilon G'(q) = 1$$



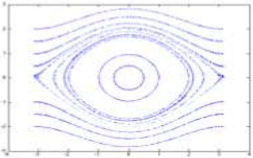
Péndulo perturbado

$$\dot{x} = y$$
$$\dot{y} = -\omega^2 \sin(x) + \epsilon \cos(\omega t)$$

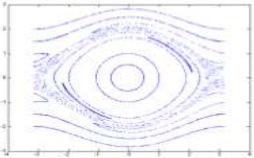




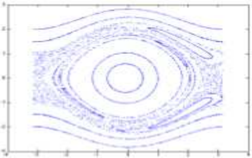
El espacio fase se “destruye”.



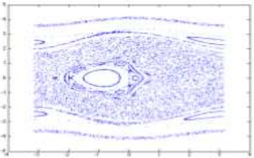
(a) $\epsilon = 0$



(b) $\epsilon = 0{,}01$



(c) $\epsilon = 0{,}1$



(d) $\epsilon = 1{,}0$

Objetos invariantes en mapeos

$\phi_{t,t_0}(x)$ $\quad x \in (q,p)$

- Puntos fijos

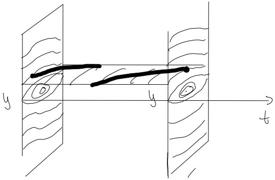
$$f(x) = x$$

- Órbitas periódicas

$$f^n(x) = x$$

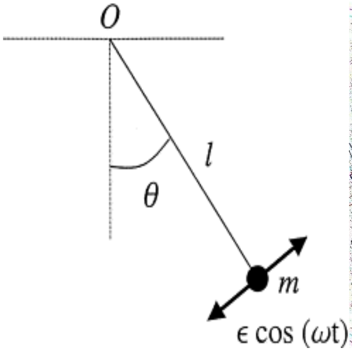
- Círculos invariantes

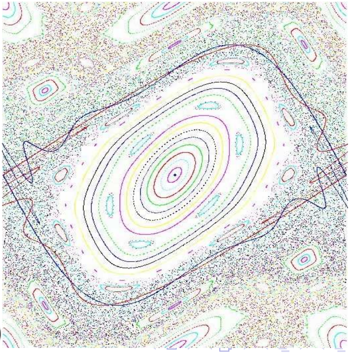
$$f(\text{círculo}) = \text{círculo}$$



Péndulo perturbado

$$H(y,x) = \frac{1}{2}y^2 + \varepsilon V(x) \times \sum_{n=-\infty}^{\infty} \delta\left(\frac{t}{T} - n\right)$$

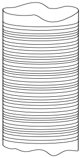




Puntos fijos, órbitas periódicas y círculos invariantes

Mapeo integrable.

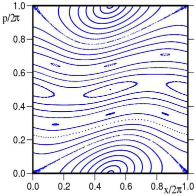
$$\begin{aligned} y' &= y \\ x' &= x + \omega(y) \pmod 1 \end{aligned}$$



ω es el número de rotación.

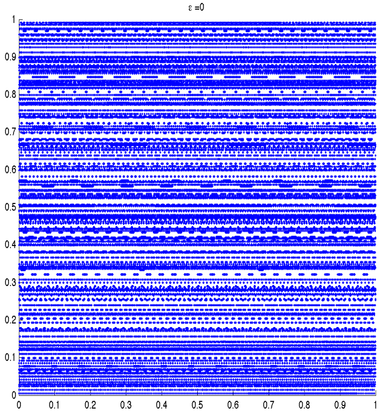
Mapeo estándar.

$$\begin{aligned} y' &= y + \frac{\varepsilon}{2\pi}\sin(2\pi x) \\ x' &= x + y' \pmod 1 \end{aligned}$$

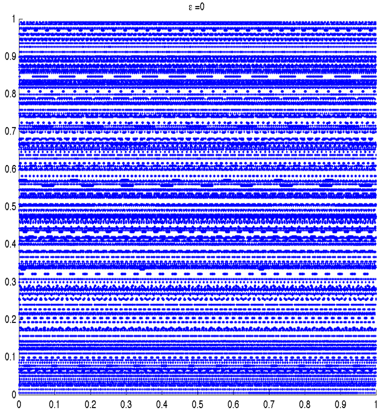


Moviendo ε

$$y' = y + \frac{\varepsilon}{2\pi}\sin(2\pi x)$$
$$x' = x + y' \pmod 1$$



$$y' = y + \frac{\varepsilon}{2\pi}\sin(2\pi x)$$
$$x' = x + y' \pmod 1$$



$$y' = y + \frac{\varepsilon}{2\pi}\sin(2\pi x)$$
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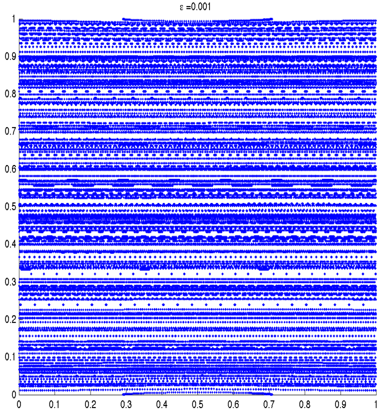
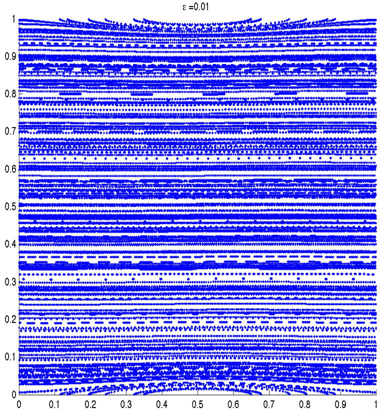


$$y' = y + \frac{\varepsilon}{2\pi} \sin(2\pi x)$$
$$x' = x + y' \pmod 1$$



$$y' = y + \frac{\varepsilon}{2\pi}\sin(2\pi x)$$
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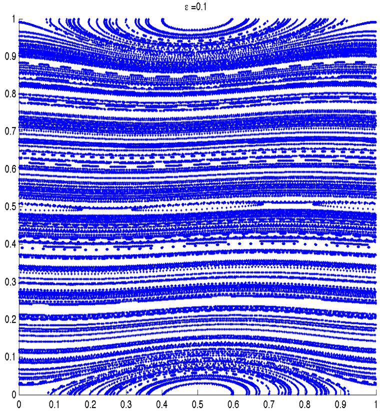

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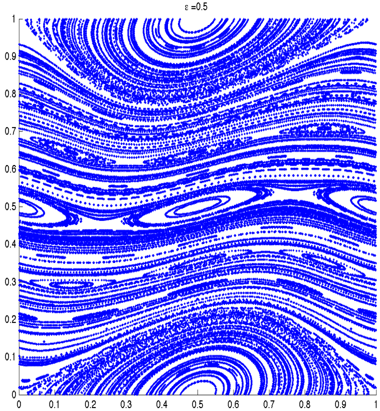

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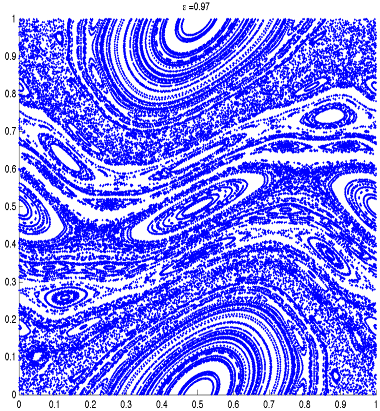

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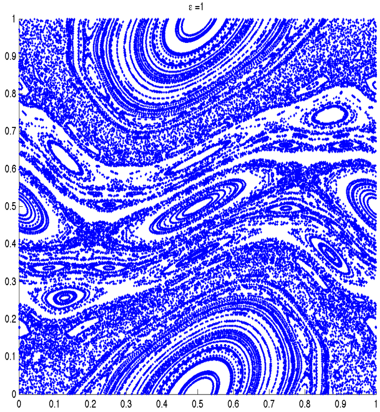

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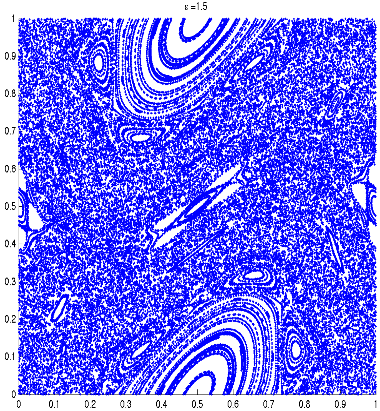

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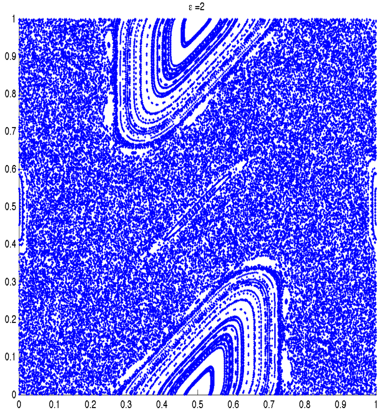

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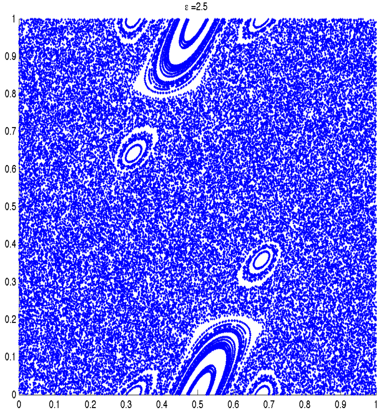

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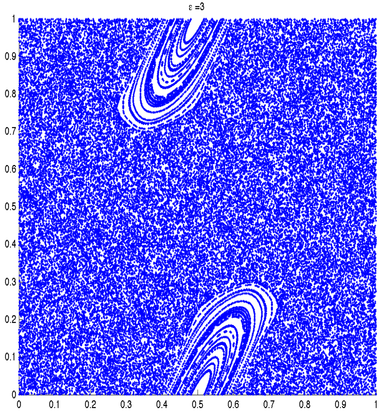

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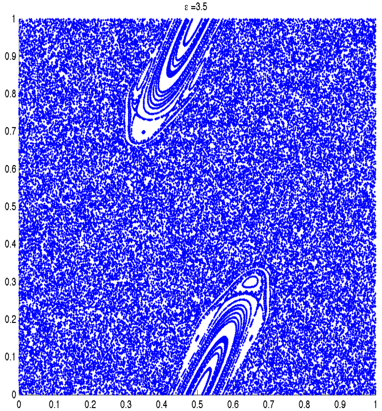

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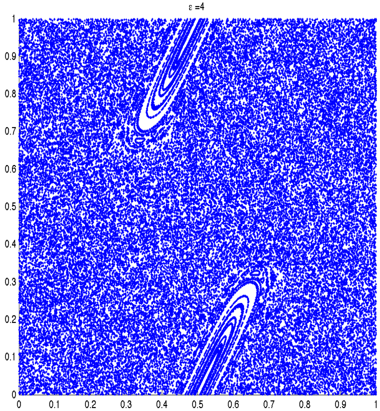

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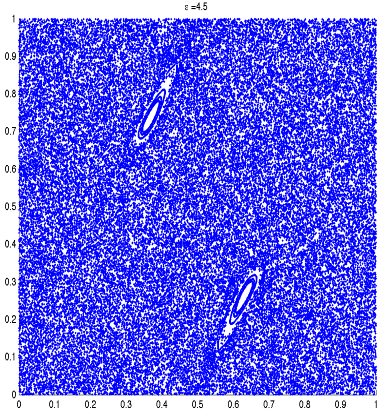

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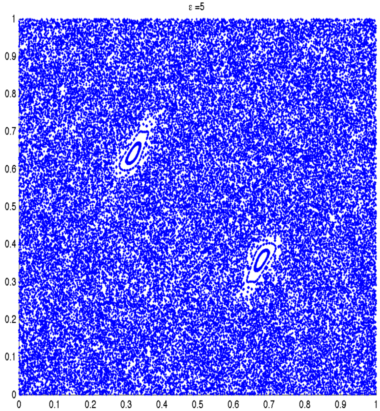

$$y' = y + \frac{\varepsilon}{2\pi} \sin(2\pi x)$$
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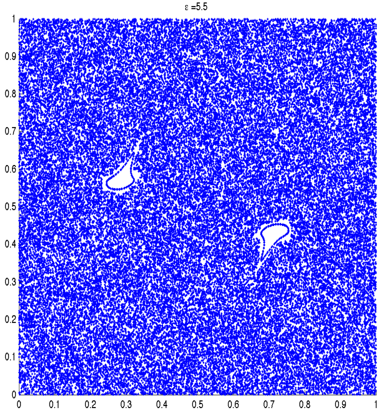

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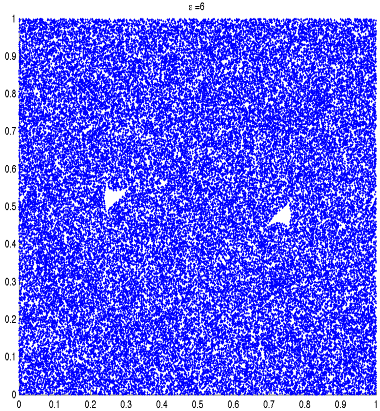
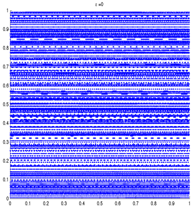
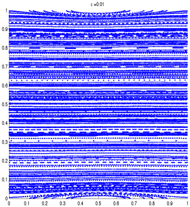


Teorema KAM

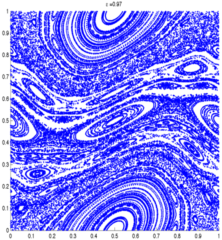
Las órbitas quasi-periódicas persisten cerca del caso integrable

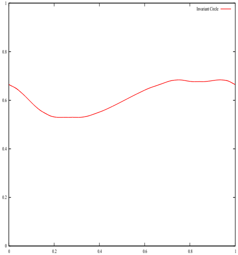




Teorema [C. - de la Llave]

Las órbitas quasi-periódicas persisten lejos del caso integrable. Desaparecen cuando la parametrización pierde regularidad (de Sobolev).





Además la prueba del teorema sugiere algoritmos muy eficientes ($\mathcal{O}(N \log N)$ operaciones).