Ejemplo 2

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij}(q) v_i v_j \quad , \qquad q \in \mathbb{R}^n$$

La matriz $g(q) = [g_{ij}(q)]$ positiva definida, simétrica.

(En el ejemplo anterior tendríamos $\frac{1}{2} m I$)

$$p_K = \frac{\partial \mathcal{L}}{\partial v_K} = \frac{1}{2}\sum_{i=1}^{n} g_{iK}(q) v_i + \frac{1}{2}\sum_{j=1}^{n} g_{Kj}(q) v_j$$

$$= \sum_{j=1}^{n} g_{Kj}(q) v_j \quad \text{(ya que } g(q) \text{ es simétrica)}$$

En notación matricial

$$p = [g(q)]v \Rightarrow v(q,p) = [g(q)]^{-1}p \Rightarrow v(q,p)^T = p^T([g(q)]^{-1})^T$$

$$H(q,p) = \sum_{i=1}^{n} p_i v_i(q,p) - \mathcal{L}(q, v(q,p)) = p^T v(q,p) - \frac{1}{2} v(q,p)^T [g(q)] v(q,p)$$

$$\mathcal{L} = \frac{1}{2} v^T [g(q)] v$$

$$= p^T [g(q)]^{-1} p - \frac{1}{2} p^T ([g(q)]^{-1})^T \cancel{[g(q)]} \cancel{[g(q)]^{-1}} p$$

$$= p^T [g(q)]^{-1} p - \frac{1}{2} p^T ([g(q)]^{-1})^T p$$

Resulta que como g es simétrica entonces g^{-1} también es simétrica.

$$H(q,p) = \frac{1}{2} p^T [g(q)]^{-1} p.$$

La ecuación de Hamilton

$(q,p) \in \mathbb{R}^{2n}$

$H = H(q,p,t)$

$$\dot{z} = J \nabla_z H(z,t) \quad , \quad z = (q,p)$$

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \quad , \quad I_n = id = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & 0 \\ 0 & 0 & & 0 & 1 \end{pmatrix}_{n \times n}$$

El corchete de Poisson asociado a J

$$[\cdot,\cdot] : C^\infty(\mathbb{R}^{2n}) \times C^\infty(\mathbb{R}^{2n}) \to C^\infty(\mathbb{R}^{2n})$$

$C^\infty(\mathbb{R}^{2n})$ son funciones C^∞ de $\mathbb{R}^{2n}$ en $\mathbb{R}$

$f, g \in C^\infty(\mathbb{R}^{2n})$

$$[f,g] = \underbrace{(\nabla_z f)^T}_{\text{vector de } 1\times 2n} \underbrace{J}_{\text{matriz de } 2n\times 2n} \underbrace{(\nabla_z g)}_{\text{vector de } 2n\times 1} \in C^\infty(\mathbb{R}^{2n})$$

Ej $\quad n=1 \quad \nabla_z = \begin{pmatrix} \frac{\partial}{\partial q} \\ \frac{\partial}{\partial p} \end{pmatrix}$

$$f, g \in C^\infty(\mathbb{R}^2) \qquad \begin{matrix} f(p,q) \\ g(p,q) \end{matrix}$$

$$\begin{aligned}
[f,g] &= (\nabla_z f)^T \; J \; (\nabla_z g) \\
&= \left(\frac{\partial f}{\partial q}, \frac{\partial f}{\partial p} \right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial g}{\partial q} \\ \frac{\partial g}{\partial p} \end{pmatrix} \\
&= \left(-\frac{\partial f}{\partial p}, \frac{\partial f}{\partial q} \right) \begin{pmatrix} \frac{\partial g}{\partial q} \\ \frac{\partial g}{\partial p} \end{pmatrix} \\
&= \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}
\end{aligned}$$

En $\mathbb{R}^{2n}$

En $\mathbb{R}^{2n}$ $\quad \nabla_z = \begin{pmatrix} \nabla_q \\ \nabla_p \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial q_1} \\ \vdots \\ \frac{\partial}{\partial q_n} \\ \frac{\partial}{\partial p_1} \\ \vdots \\ \frac{\partial}{\partial p_n} \end{pmatrix}$

$$f(q,p) = f(q_1, \dots, q_n, p_1, \dots, p_n)$$

$$g(q,p) = g(\_\_\_\_, \_\_\_\_)$$

$$\begin{aligned}
[f,g](z) &= (\nabla_z f(z))^T \; J \; \nabla_z g(z) \\
&= \left((\nabla_q f)^T, (\nabla_p f)^T \right) \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} \nabla_q g \\ \nabla_p g \end{pmatrix} \\
&= \left((\nabla_q f)^T, (\nabla_p f)^T \right) \begin{pmatrix} \nabla_p g \\ -\nabla_q g \end{pmatrix} = \\
&= (\nabla_q f)^T \nabla_p g - (\nabla_p f)^T \nabla_q g = \sum_{i=1}^n \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial p_i} \right)
\end{aligned}$$

Proposición

Sea $f:\mathbb{R}^{2n}\to\mathbb{R}$, $f\in C^\infty(\mathbb{R}^{2n})$ y

$q(t), p(t)$ solución de
$$\dot{q}_i(t) = \frac{\partial H}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Entonces $\dot{f} = [f, H]$.

Dem $\quad f(z) = f(q,p)$

$$\frac{df}{dt}(q(t),p(t)) = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\cdot\dot{p}_i\right) = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right)$$
$$= [f, H] \quad //$$

Corolario

$f:\mathbb{R}^{2n}\to\mathbb{R}$, $C^\infty(\mathbb{R}^{2n})$ es una constante de movimento $(\dot{f}=0)$ de $\dot{z} = J\nabla H(z)$

$\iff [f, H] = 0$

Proposición

El corchete de Poisson satisface las siguientes propiedades.

(i) $[f,g] = -[g,f]$, $\forall f,g\in C^\infty(\mathbb{R}^{2n})$ (antisimetría)

(ii) $[f+g,h] = [f,h] + [g,h]$, $[\lambda f, g] = \lambda[f,g]$ donde $f,g,h\in C^\infty(\mathbb{R}^{2n})$, $\lambda\in\mathbb{R}$. (bilinealidad)

(iii) $[[f,g],h] + [[h,f],g] + [[g,h],f] = 0$ $\forall f,g,h\in C^\infty(\mathbb{R}^{2n})$, (Propiedad de Jacobi)

(iv) $[fg, h] = f[g,h] + [f,h]g$, $\forall f,g,h\in C^\infty(\mathbb{R}^{2n})$ (regla de Leibnitz) (es una derivación)

Dem

(i)
$$[f,g] = \nabla_q f^T\nabla_p g - \nabla_p f^T\nabla_q g$$
$$= -\left(\nabla_p g^T\nabla_q f + \nabla_q g^T\nabla_p f\right)$$
$$= -[g,f]$$

(ii) por la linealidad de la derivada.

(iv)
$$[fg,h] = \left(\nabla_q(fg)\right)^T\nabla_p h - \left(\nabla_p(fg)\right)^T\nabla_q h$$
$$= \left(\left(\nabla_q f\right)^T g + f\left(\nabla_q g\right)^T\right)h - \left(\left(\nabla_p f\right)^T g + f\left(\nabla_p g\right)^T\right)\nabla_q h$$

$$= \left((\nabla_q f)^T g + f (\nabla_q g)^T \right) \nabla_p h - \left((\nabla_p f)^T g + f (\nabla_p g)^T \right) \nabla_q h$$

$$= \left[(\nabla_q f)^T \nabla_p h - (\nabla_p f)^T \nabla_q h \right] g + f \left[(\nabla_q g)^T \nabla_p h - (\nabla_p g)^T \nabla_q h \right]$$

$$= [f, h] g + f [g, h] .$$

//

(iii) $[[f, g], h] + [[h, f], g] + [[g, h], f] = 0$