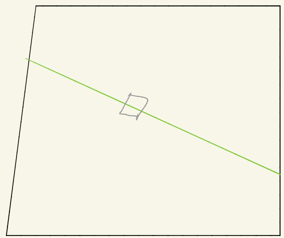
Ejemplo con tiempo

$$f(x,y,t) = y + (1+\varepsilon \sin \omega t)x \quad , \quad \varepsilon \geq 0 \text{ pequeña.}$$

$$f(x,y,t) = 0$$



$$\nabla f(x,y,t)$$

$$\lambda = \lambda(t)$$

$$y = h(x,t) \quad \text{TFI}$$

$$y = -(1+\varepsilon \sin \omega t)x$$

$$\dot{y} = \tilde{h}(x,\dot{x},t)$$

$$\ddot{y} = \tilde{\tilde{h}}(x,\dot{x},\ddot{x},t)$$

$$\ddot{x} = \tilde{f}(\quad \dots \quad)$$

Caso de varios contornos

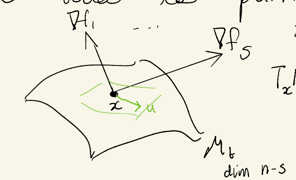
$$m\ddot{x} = F, \quad x \in \mathbb{R}^n$$

$$f_j(x,t) = 0, \quad j = 1,\dots,s < n$$

$\hookrightarrow$ variedad $\mathcal{M}_t$ $\dim \mathcal{M}_t = n-s$

$$m\ddot{x} = F + \sum_{j=1}^{s} \lambda_j \nabla f_j(x,t)$$

$\rightarrow$ Necesitamos que $\{\nabla f_j\}_{j=1}^{s}$ sean linealmente independientes sobre todos los puntos de la variedad.



Estos campos vectoriales

$$T_x\mathbb{R}^n = \mathbb{R}^n = \underbrace{T_x\mathcal{M}_t}_{(n-s)\text{-dim}} \oplus \underbrace{\operatorname{span}\{\nabla f_j : j=1,\dots,s\}}_{\dim s}$$

$$= T_x\mathcal{M}_t \oplus (T_x\mathcal{M}_t)^{\perp}$$

$$\mathcal{M}_t = \{x \in \mathbb{R}^n : f_1(x,t) = 0, \dots, f_s(x,t) = 0\}$$

$$\dim \mathcal{M}_t = n-s$$

Sea $u \in T_x\mathcal{M}_t$ campo vectorial tangente a $\mathcal{M}_t$

tenemes que $\langle \nabla f_j, u \rangle = 0, \quad \forall j = 1,\dots,s$

d'Alembert

$$\langle m\ddot{x} - F, u \rangle = 0$$

Solución $x(t)$

$$\lambda_j = |\nabla f|^{-2} \langle m\ddot{x} - F, \nabla f_j \rangle$$

Formulación Lagrangiana I

$x\in\mathbb{R}^n$, Lagrangiana $\mathcal{L}: \underbrace{T\mathbb{R}^n}_{\mathbb{R}^n\times\mathbb{R}^n} \times \mathbb{R} \longrightarrow \mathbb{R}$

s contornos $f_j(x,t)=0$, $j=1,\dots,s$

$$\tilde{\mathcal{L}}(x,\dot{x},t) = \mathcal{L}(x,\dot{x},t) + \sum_{j=1}^{s} \lambda_j f_j(x,t)$$

¿Cómo se ven las ecs. E-L?

$$\frac{d}{dt}\left(\frac{\partial\tilde{\mathcal{L}}}{\partial \dot{x}_i}\right) - \frac{\partial \tilde{\mathcal{L}}}{\partial x_i} = 0, \quad i=1,\dots,n$$

$$\iff \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial \dot{x}_i}\right) - \frac{\partial \mathcal{L}}{\partial x_i} - \sum_{j=1}^{s} \lambda_j \frac{\partial f_j}{\partial x_i} = 0, \quad i=1,\dots,n$$

$$\iff \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial \dot{x}_i}\right) - \frac{\partial \mathcal{L}}{\partial x_i} = \sum_{j=1}^{s} \lambda_j \frac{\partial f_j}{\partial x_i}$$

En mec. $\rightarrow$ hemos añadido fuerzas $\sum_{j=1}^{s} \lambda_j \nabla f_j$

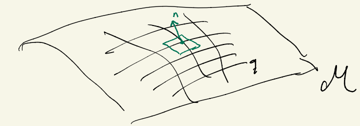
En sistemas Lagrangianos, añadir términos $\sum_j \lambda_j f_j$ al Lagrangiano es equivalente a la teoría de contornos ideales.

Formulación alternativa

$$\{ f_j(x) = 0,\ j=1,\dots,s \} = \mathcal{M}$$

variedad en $\mathbb{R}^n$ (dim $n-s$)

$\mathbb{R}^n$



$\dim \mathcal{M} = n-s$

q – coordenadas en $\mathcal{M}$ dim $n-s$

Sean n – coordenadas del espacio normal. de dimensión s

Tenemos (q,n) coordenadas de $\mathcal{M}\times\mathbb{R}^s = \mathbb{R}^n = T_x\mathbb{R}^n$