Ejemplo 1

$$\mathcal{L} = \frac{1}{2} m |\dot q|^2 - U(q), \quad q \in \mathbb{R}^2$$

q: coords Euclidianas $\quad q = (q_1, q_2)$

$$|\dot q|^2 = \dot q_1^2 + \dot q_2^2$$

E-L

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0, \qquad \frac{d}{dt} m \dot q_1 - \frac{\partial U}{\partial q_1} = 0$$

$$\frac{d}{dt} m \dot q_2 - \frac{\partial U}{\partial q_2} = 0$$

$s =$ coords polares $\qquad s = [s_1, s_2] = [r, \theta]$

$\tilde{\mathcal{L}} = \mathcal{L}$ en coordenadas polares

$$\tilde{\mathcal{L}}(r, \theta, \dot r, \dot\theta) = \mathcal{L}(q_1(r,\theta), q_2(r,\theta), \dot q_1(r,\theta), \dot q_2(r,\theta))$$

$$q_1 = r\cos\theta$$

$$q_2 = r\sin\theta$$

Vamos a escribir las ecuaciones de E-L para $\tilde{\mathcal{L}}$

$$\frac{d}{dt}\frac{\partial \tilde{\mathcal{L}}}{\partial \dot r} - \frac{\partial \tilde{\mathcal{L}}}{\partial r} = 0$$

$$\frac{d}{dt}\frac{\partial \tilde{\mathcal{L}}}{\partial \dot \theta} - \frac{\partial \tilde{\mathcal{L}}}{\partial \theta} = 0$$

$$\tilde{\mathcal{L}} = \frac{1}{2} m \Big[\underbrace{\Big(\frac{d}{dt} r\cos\theta\Big)^2}_{(\dot q_1)^2} + \Big(\frac{d}{dt} r\sin\theta\Big)^2 \Big] - U(q_1(r,\theta), q_2(r,\theta))$$

$$= \frac{1}{2} m \big[(\dot r\cos\theta - r\dot\theta\sin\theta)^2 + (\dot r\sin\theta + r\dot\theta\cos\theta)^2 \big] - U(q_1(r,\theta), q_2(r,\theta))$$

$$= \frac{1}{2} m \Big[\dot r^2\cos^2\theta - \cancel{2 r\dot r\dot\theta\cos\theta\sin\theta} + r^2\dot\theta^2\sin^2\theta + \dot r^2\sin^2\theta + \cancel{2 r\dot r\dot\theta\cos\theta\sin\theta} + r^2\dot\theta^2\cos^2\theta \Big] - U(q_1(r,\theta), q_2(r,\theta)) = \frac{1}{2} m (\dot r^2 + r^2\dot\theta^2) - U(r,\theta)$$

Simplificación

$$\tilde{\mathcal{L}} = \frac{1}{2} m (\dot r^2 + r^2\dot\theta^2) - U(r)$$

Sea $\quad U(q) = U(|q|) = U(r)$

E-L

(r) $\quad \frac{d}{dt} m\dot r - m r\dot\theta^2 + U'(r) = 0$

(θ) $\quad \frac{d}{dt} r^2\dot\theta = 0 \implies r^2\dot\theta = I_1$ - constante.

$$\implies m\ddot r = m\frac{r^4\dot\theta^2}{r^3} - U'(r) = m\frac{I_1^2}{r^3} - U'(r)$$

Tenemos un principio de invarianza con las ecuaciones de Euler-Lagrange

"El principio de acción estacionaria no depende del sistema de coordenadas"

Simetrías y "Leyes de conservación"

Sistema Lagrangiano $(M, \mathcal{L}) \to$ evolución : ec. E-L.

$$x = (q, v) \subset TM \qquad \dot x = F(x, t)$$

q: posición, v: velocidad.

$$x(0) = [q(0), v(0)], \quad v(0) = \dot q(0)$$

Cantidad conservada $\quad J : TM \times \mathbb{R} \to \mathbb{R}$

tal que

$$\frac{d}{dt} J(q(t), v(t), t) = 0, \quad \forall (q(t), v(t)) \text{ soluciones de ec. E-L}$$

$$M = \mathbb{R}^n, \quad \mathcal{L} = \frac{1}{2}|\dot q|^2 - U(q)$$

$$E = \frac{1}{2}|\dot q|^2 + U(q) \rightsquigarrow \text{Energía} \qquad \frac{d}{dt}E = 0$$

E es una cantidad conservada.

Sea $q \times \dot q$ "momento angular".